MATH 320, Spring SEMESTER 2009

Final Exam

Name (print): ____________________________

Signature: ____________________________

INSTRUCTIONS

(1) Write the answers on your own paper.
(2) Answers with factorials or powers are accepted, e.g., 10!, $9 \times 8 \times 7$ or $100 - 2^4$ are acceptable.
(3) Present your solutions in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.
(4) You can search the Internet for solutions, but you cannot post the question to the Internet for answers.
(5) You can work as groups for the answers, but each one of you must write the answer in your own words. If any two of the answers are written in strikingly similar way, then both will be counted as cheating.

(1) (10 points) A professor tells three jokes in his class each year. What is the minimal number of jokes that he needs in order never to repeat the exact same triple of jokes (without order) over a period of 12 years?

(2) (10 points) Consider all the positive integers *less than* a million, how many of them contain the digit 2 but not 4?

(3) (10 points) There are 5 people sitting around a poker table, including you. Everyone is dealt 2 cards facing down, and you have the Ace of spades and the 3 of hearts. What is the probability that there is *exactly* one Ace dealt other than yours?

(4) (10 points) A group of n couples sits around a circular table for a party. In how many ways can the group be seated so that no husband and wife sit together?

(5) Let $\{F_n\}$ be the Fibonacci numbers. Prove that the sequence $\{F_n \mod m\}$ is eventually periodic for any integer $m > 0$ by completing the following steps.
   (a) (10 points) Prove that for any integer $m > 0$, there always exist integers $0 < r < s$ such that $F_r = F_s \pmod{m}$ and $F_{r+1} = F_{s+1} \pmod{m}$.
   (b) (10 points) Prove that if $F_r = F_s \pmod{m}$ and $F_{r+1} = F_{s+1} \pmod{m}$, with $0 < r < s$, then $F_{u+r} = F_{u+s} \pmod{m}$ for all $u \geq 0$.
   (c) (5 points) If the case above happens, $\{F_n \mod m\}$ is periodic for $n \geq r$.
   (Notation: $a \mod p$ is the remainder of a divided by p, and $a = b \pmod{p}$ means the remainders of a and b are the same when divided by p.)

(6) Prove that $\lfloor (2+\sqrt{3})^n \rfloor$ is odd for any n by completing the following steps.
   (a) (10 points) Find the explicit formula to the recursion $h_n = 4h_{n-1} - h_{n-2}$, $h_0 = 2, h_1 = 4$.
   (b) (10 points) Show that h_n is even for any n based on the recursion above.
   (c) (5 points) Prove that $\lfloor (2+\sqrt{3})^n \rfloor = h_n - 1$, and therefore is odd.

(7) (10 points) Let $2n$ equally spaced points be chosen on a circle. Let h_n denote the number of ways to join those points in pairs so that the resulting line segments do not intersect. Prove that h_n is the Catalan number.