Midterm Exam 1 Solutions

Problem 1: Potpourri (20 pts)

(a) (4 pts) Let X be a discrete random variable with mean $\mu = E(X)$. Show the following formula

$$E(X-\mu)^2 = E(X^2) - \mu^2.$$

Solution: By linearity of expectation

$$\begin{aligned} E(X-\mu)^2 &= E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2. \end{aligned}$$

(b) (8 pts) Let X be a discrete random variable with mean $\mu = E(X)$ and variance $\sigma^2 = \text{Var(X)}$. Show the following formula

$$E(X-\mu)^3 = E(X^3) - 3\mu\sigma^2 - \mu^3.$$

Solution: Let's expand the cube

$$\begin{aligned} (X-\mu)^3 &= X^3 + \binom{3}{1}X^2(-\mu) + \binom{3}{2}X(-\mu)^2 + (-\mu)^3 \\ &= X^3 - 3X^2\mu + 3X\mu^2 - \mu^3 \end{aligned}$$

Using Linearity of expectation and the fact that (by part a) $E(X^2) = \sigma^2 + \mu^2$

$$\begin{aligned} E(X-\mu)^3 &= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3 \\ &= E(X^3) - 3\mu(\sigma^2 + \mu^2) + 3\mu^3 - \mu^3 \\ &= E(X^3) - 3\mu\sigma^2 - \mu^3. \end{aligned}$$

(c) (8 pts) Suppose $X_1, X_2, X_3 \ldots, X_n$ are independent random variables with a common mean $E(X_i) = \mu$ and variance $\text{Var}(X_i) = \sigma^2 > 0$. Let

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

be the sample mean. Show that

$$P(|\overline{X}_n - \mu| > \sigma) \leq \frac{1}{n}.$$

Solution: By linearity of expected value

$$E(\overline{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n\mu}{n} = \mu,$$

and using the fact that $X_1, X_2, \ldots X_n$ are independent

$$\text{Var}(\overline{X}_n) = \frac{1}{n^2}\text{Var}(X_1 + \ldots + X_n) = \frac{n\sigma}{n^2} = \frac{\sigma}{n}.$$

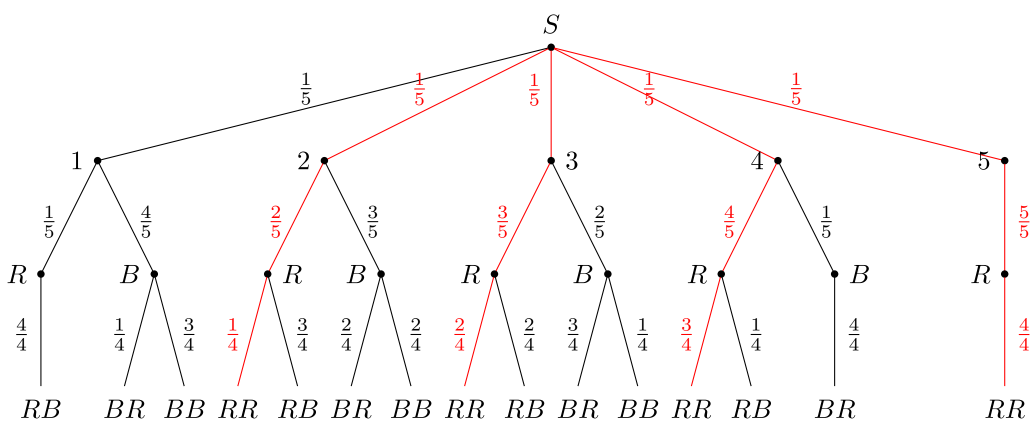
Therefore, by Chebyshev

$$P(|\overline{X}_n - \mu| > \sigma) \leq \frac{\text{Var}(\overline{X}_n)}{\sigma} \leq \frac{1}{n}.$$

Problem 2: One fish two fish (18 pts)
Suppose you work at a pet store and have five identical fish bowls. Inside each bowl are 5 fish, comprised of a mix of red and blue fish. The bowls are labeled $1, 2, 3, 4, 5$ indicating the number of red fish in each bowl (e.g. bowl 2 has exactly two red fish). A customer wants two red fish. Since you are in a rush, you pick a bowl at random and pick out two fish from that bowl (also at random).

(a) (8pts) What is the probability that both fish you selected are red?

Solution: Lets draw the tree diagram.



Therefore

$$P(R) = \left(\frac{1}{4}\right)\left(\frac{2}{5}\right)\left(\frac{1}{5}\right) + \left(\frac{2}{4}\right)\left(\frac{3}{5}\right)\left(\frac{1}{5}\right) + \left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right) + \left(\frac{4}{4}\right)\left(\frac{5}{5}\right)\left(\frac{1}{5}\right)$$
$$= \frac{(2 + 6 + 12 + 20)}{20(5)} = \frac{40}{20(5)} = \frac{2}{5}.$$

(b) (10pts) You are in luck! Both fish are red. What is the probability that you selected bowl 3?

Solution: We want to compute the probabilty $P(3|RR)$. Using Bayes, we find,

$$P(3|RR) = \frac{P(RR|3)P(3)}{P(RR)} = \frac{\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{5}\right)}{\frac{2}{5}} = \frac{3}{20}.$$

Problem 3: The mannersbit (25 pts)
Suppose a donut is left out at the office in the morning for anyone who wants to eat it. Being too polite to eat everything that's left out, each person who comes across it eats only half of what's left, leaving the remaining half for the next person. Suppose that number people who come across the donut in a given work day is Poisson distributed with an average rate of 10 people per work day.

(a) (8 pts) Assuming an 8 hour work day, what is the probability of exactly one person coming by the donut in a given hour?

Solution: We first need to convert the rate to hours. If on average 10 people come per work day, then $\lambda = 10/8 = \frac{5}{4}$ people come per hour. Let $X \sim \text{Poisson}(5/4)$. The probability of 1 person coming by is just

$$P(X=1) = \frac{\left(\frac{5}{4}\right)^1}{1!}e^{-\frac{5}{4}} = \frac{5}{4}e^{-\frac{5}{4}}.$$

(b) (5 pts) If exactly n people come across the donut in a given work day (all at different times) what fraction of the donut is left at the end of the work day?

Solution: 2^{-n} of the donut will be the fraction left.

(c) (12pts) What is the expected fraction of the donut left over at the end of a given work day? (For full credit, give a simple answer that is not an infinite series)

Solution: Let $Y \sim \text{Poisson}(10)$ be the number of people who come by the donut in a given work day. The fraction of donut at the end of the work day is 2^{-Y}. Therefore the expected fraction left will be

$$E(2^{-Y}) = \sum_{k=0}^{\infty} 2^{-k}P(Y=k) = \sum_{k=0}^{\infty} 2^{-k}\frac{(10)^k}{k!}e^{-10}$$
$$= \sum_{k=0}^{\infty} \frac{\left(\frac{10}{2}\right)^k}{k!}e^{-10} = e^{\frac{10}{2}-10} = e^{-5}.$$

Problem 4: Exams of the round table (25 pts)
Suppose there are 20 students seated at a round table to take an exam. The professor has made 4 different exams (5 of each type) and distributes them randomly among the students (so that each arrangement is equally likely).

(a) (5 pts) How many unique ways are there to distribute the different exam types among the students?

Solution: There are 20 students and assigning them different exam type is the same as partitioning 20 students into 4 groups of 5 students each. This is just the multinomial coefficient

$$\binom{20}{5\,5\,5\,5}.$$

(b) (8 pts) What is the probability that a given student is seated next to someone with the same exam type?

Solution: For a given person (lets call them you), there are two people seated on either side of you. We want to know the probability that at least one of those people has the same exam as that as you, call that event A. In this case it is easier to think of the complement event $\overline{A}$, namely the event that you are seated next to two people, neither of them has the same exam as you. Define the following events

$$L = \text{the person on the left has a different exam than you}$$

$$R = \text{the person on the right has a different exam than you.}$$

Then $\overline{A} = L \cap R$. We can calculate the probability by the law of multiplication,

$$P(L \cap R) = P(L|R)P(R).$$

For the person on the right, there are only 15 possible exams out the 19 that are not the same as yours. Therefore

$$P(R) = \frac{15}{19}.$$

Likewise, given that the person on the right doesn't have the same exam as you, there are only 14 possible remaining exams out of 18 that are not the same as yours. Therefore

$$P(L|R) = \frac{14}{18}.$$

It follows that

$$P(L \cap R) = \left(\frac{14}{18}\right)\left(\frac{15}{19}\right) = \frac{14(15)}{18(19)} = \frac{7(5)}{3(19)} = \frac{35}{57}.$$

and so

$$P(A) = 1 - \frac{35}{57} = \frac{22}{57}.$$

(c) (12 pts) What is the expected number of students who are seated next to someone with the same exam type?

Solution: Let X_i be a Bernoulli random variable define by

$$X_i = \begin{cases} 1 & \text{if person i is seated next to a person with the same exam} \\ 0 & \text{otherwise} \end{cases}$$

The number of people who are seated next to someone with the same exam type is

$$X = \sum_{i=1}^{20} X_i$$

We know by part b) that

$$E(X_i) = \frac{22}{57}.$$

Therefore by linearity of expected value

$$E(X) = 20\left(\frac{22}{57}\right) = \frac{20(22)}{57}.$$

Problem 5: Odd one out (12 pts)
A group of 16 students (4 ECON and 12 APMA) are randomly divided into 4 study groups of size 4.

1. (4 pts) How many ways are there to divide all the students into the four study groups?

   Solution: If you view the groups as distinct, this is just the multinomial coefficient

   $$\binom{16}{4\,4\,4\,4}$$

   If you don't view the groups are being distinct then you must divide by all the ways to permute 4 groups (4!), this is just

   $$\binom{16}{4\,4\,4\,4}/4!$$

   Either answer is acceptable.

2. (8 pts) What is the probability that each study group includes an ECON student?

   Solution: The answer does not depend on whether you view the groups as distinct are not, because once the ECON students are assigned, the groups are now distinct.

   Lets consider the case that the groups are distinct and see how many way that each group contains an ECON studnet. Since there are only 4 ECON students, one must be assigned to each group, there are 4! ways to do that. That leaves only 3 spots left in each group for the APMA students. This is the same as partitioning 12 students into 4 groups of size 3. There are

   $$\binom{12}{3\,3\,3\,3}$$

   ways to do that. Therefore the probability is

   $$\frac{4!\binom{12}{3\,3\,3\,3}}{\binom{16}{4\,4\,4\,4}} = \frac{(4!)(4^4)}{(16)(15)(14)(13)} = \frac{2^5}{5(7)(13)}.$$