Midterm 1 Practice Problem Solutions

These are a collection of practice problems for the first midterm exam. Some of the problems are more computationally involved than those on the exam will be. The actual exam will of course be much shorter than this. If you can do these problems (without looking at solutions), there is a high probability that you will do well on the exam.

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1. How many unique arrangements are there of the word PROBABILITY?

Solution: There are 11 letters total with 1 of each of the letters P,R,O,A,L,T,Y and 2 of each of the letters B,I. This can be viewed as a number of partitions of the blank word (with 11 blanks)

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into 7 bins of size 1 corresponding to where the letters P,R,O,A,L,T,Y go and 2 bins of size two corresponding to where you put the letters P and B each. This is given by the multinomial coefficient

$$\binom{11}{1\,1\,1\,1\,1\,1\,1\,2\,2} = 9979200.$$

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2. Suppose that who events A and B satisfy $P(A) = 0.4$, $P(B) = 0.3$ and $P(\overline{(A \cup B)}) = 0.42$. Are A and B independent?

Solution: We have $P(A \cup B) = 1 - 0.42 = 0.58$ and we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Therefore

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.3 - 0.58 = 0.12$$

Since this is the same as

$$P(A)P(B) = (0.4)(0.3) = 0.12,$$

we conclude they are independent.

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3. There are six men and seven women in a ballroom dancing class. If four men and four women are chosen and paired off, how many pairings are possible?

Solution: First we choose two groups of 4. There are $\binom{6}{4}$ ways to choose the men and $\binom{7}{4}$ ways to choose the women. Once we have these two groups of 4, we can ask how many ways there are to pair off each. This is just 4! since for a given person and a group (men or women) there are 4 ways to choose a partner, once they are chosen, there are 3 ways to choose a partner for the second person and so on. This means there are a total of

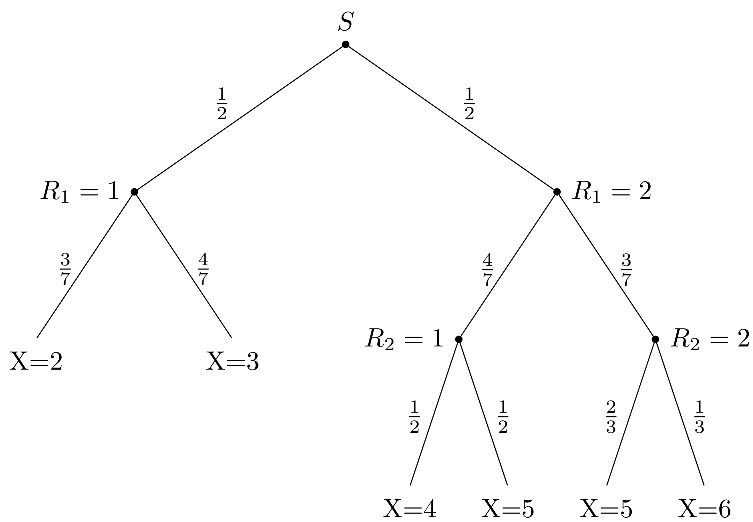
$$\binom{6}{4}\binom{7}{4}4! \quad \text{possible pairings.}$$

4. There are 8 cards in a hat:

$$\{1\heartsuit, 1\spadesuit, 1\diamondsuit, 1\clubsuit, 2\heartsuit, 2\spadesuit, 2\diamondsuit, 2\clubsuit\}$$

You draw one card at random. If its rank is 1 you draw one more card; if its rank is two you draw two more cards. Let X be the sum of the ranks on the 2 or 3 cards drawn. Find E(X).

Solution: Let R_1, R_2 denote the rank of the first and second card draws and X be the sum of the two or three card that results at the end. The following tree shows all the possible values of X with the associate conditional probabilities.



Multiplying up the branches (realizing that $X = 5$ shows up on two branches) gives the following probability table

x	2	3	4	5	6
$p(x)$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$.

Therefore

$$E(X) = 2\left(\frac{3}{14}\right) + 3\left(\frac{2}{7}\right) + 4\left(\frac{1}{7}\right) + 5\left(\frac{2}{7}\right) + 6\left(\frac{1}{14}\right) = \frac{26}{7}$$

5. Would you rather take a multiple-choice test or a full recall test? If you have no knowledge of the test material, you will score zero on a full recall test. However, if you are given 5 choices for each multiple choice question, you have at least one chance in five of guessing each correct answer! Suppose that a multiple choice exam contains 100 questions, each with 5 possible answers, and you guess the answer to each of the questions.

(a) What is the expected value of the number Y of questions that will be answered correctly?

(b) Find the standard deviation of Y.

(c) Calculate the intervals $\mu \pm 2\sigma$ and $\mu \pm 3\sigma$.

(d) If the results of the exam are curved so that 50 correct answers is a passing score, are you likely to receive a passing score? (What does Chebyshevs inequality give you?)

Solution:

(a) Here Y is binomial with $p = \frac{1}{5}$ and $n = 100$. Therefore $E(Y) = np = 20$

(b) The standard deviation is just $\sigma = \sqrt{np(1-p)} = \sqrt{16} = 4$.

(c) The intervals are $(12, 28)$, $(8, 32)$.

(d) We don't have a table that allows us to compute $P(Y \geq 50)$ very easily. So lets use Chebyshev's inequality

$$P(Y > 50) = P(Y - 20 > 30) \leq P(|Y - 20| > 30) < \frac{16}{30^2} \approx 0.0178.$$

The actual probability is even lower. Perhaps you should just study for the exam...

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6. For a bill to make it to the office of the president, it must be passed by both the House and the Senate. Suppose that of all bills, 60 percent pass the House, 80 percent pass the Senate, and 90 percent pass at least one of the two. What is the probability that the next bill will make it to the office of the president?

Solution: Let A be the event the bill passes the House the B be the event that the bill passes the Senate. We want to know $P(A \cap B)$. The problem tells us that

$$P(A) = 0.6, \quad P(B) = 0.8, \quad P(A \cup B) = 0.9.$$

Therefore

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.6 + 0.8 - 0.9 = 0.5$$

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7. A student attends a career fair and visits booths of different companies. Their chance of getting an interview with a company after they visit a booth depends on whether they took APMA 1650 or not. If the student took 1650 then the chance of being offered an interview is $p = 95\%$. If the student didn't take 1650 then they have $p = 15\%$ chance of being offered an interview.

(a) Let X record the number of booths a student visits until they receive an interview. Give the probability distribution function for X in terms of p.

(b) On average, how many booths must student who took 1650 visit before getting an interview? How about a student who hasn't?

(c) Suppose each student visits 5 booths during a career fair, what is the probability that a student who took 1650 will not get an interview? What is the probability that a student who didn't take 1650 will get an interview?

Solution:

(a) We can think of each booth visit as an independent Bernoulli random variable with probability of success p. The number of visits until an interview is just the number of tries until success (as opposed to the number of failures before success), which is a version of the Geometric(p) random variable (the version in the text). Since you must fail $n-1$ times before a success, the probability is just

$$P(X = n) = (1-p)^{n-1}p.$$

(b) The average is just (using the fact that we know the two definition of geometric random variables differ by 1)

$$E(X) = 1 + \frac{1-p}{p} = \frac{1}{p}$$

Choosing $p = 0.95$ and $p = 0.15$ gives the answer.

(c) Since a 1650 student fails doesn't get at interview at a booth with probability $q = 0.05$ the probability of failing all 5 interviews is really low

$$q^5 = (0.05)^5 \approx 0.000000313.$$

While for the students who didn't take 1650, the probability of not getting an interview at a given booth is just $q = 0.85$. Therefore the probability of getting at least one interview is just one minus the probability of not getting any interviews, i.e.

$$1 - q^5 = 1 - (0.85)^5 \approx 0.556.$$

Roughly 50-50.

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8. Suppose a pond contains two colors of fish, 6 red and 9 blue. You catch one fish in a net equally likely at random from all fish in the pond, note its color and then return the fish along with 2 more of the same color into the pond and then repeat.

(a) What is the probability that the first fish caught is Red and the next two are Blue?

(b) What is the probability that there are more Red fish than Blue fish after 3 iterations?

Solution:

(a) This is just

$$P(R_1B_2B_3) = P(B_3|B_2R_1)P(B_2|R_1)P(R_1) = \left(\frac{11}{19}\right)\left(\frac{9}{17}\right)\left(\frac{6}{15}\right)$$

(b) This can only happen if you picked red every time

$$P(R1R_2R_3) = P(R_3|R_2R_1)P(R_2|R_1)P(R_1) = \left(\frac{10}{19}\right)\left(\frac{8}{17}\right)\left(\frac{6}{15}\right)$$

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9. For two events A and B, show that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Solution: We split both $A \cup B$ and A into a disjoint unions

$$A \cup B = (A \backslash B) \cup B \quad A = (A \backslash B) \cup (A \cap B)$$

Therefore using the additive property of disjoint events

$$P(A \cup B) = P(A \backslash B) + P(B), \quad P(A) = P(A \backslash B) + P(A \cap B)$$

Subtracting the two above equations from each other, the $P(A \backslash B)'s$ cancel and we obtain

$$P(A \cup B) - P(A) = P(B) - P(A \cap B).$$

Rearranging gives the result.

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10. Suppose that $X \sim \text{Binomial}(n, 0.5)$. Find the distribution function of $Y = 2X$.

Solution: For y taking values $0, 2, \ldots, 2n$, we have

$$P(Y = y) = P(2X = y) = P(X = y/2) = \binom{n}{y/2}\left(\frac{1}{2}\right)^n.$$

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11. Let A and B be two events. Suppose the probability that neither A or B occurs is 2/3. What is the probability that one or both occur?

Solution: We are trying to find $P(A \cup B)$ and we are given $P(\overline{A} \cap \overline{B}) = \frac{2}{3}$. By DeMorgan's

$$\overline{A} \cap \overline{B} = \overline{A \cup B}.$$

Therefore

$$P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - P(\overline{A} \cap \overline{B}) = 1 - \frac{2}{3} = \frac{1}{3}$$

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12. Let C and D be two events with $P(C) = 0.25$, $P(D) = 0.45$, and $P(C \cap D) = 0.1$. What is $P(\bar{C} \cap D)$?

Solution: (It helps to draw a Venn diagram and label the associated areas) Recall that $\overline{C} \cap D = D \backslash C$. Using the law of differences

$$P(D \backslash C) = P(D) - P(D \cap C) = 0.45 - 0.1 = 0.35.$$

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13. Suppose 100 people all toss a hat into a box and then proceed to randomly pick out a hat. What is the expected number of people to get their own hat back?

Solution: Let X_i be a Bernoulli random variable which is equal to 1 if person i gets their hat and 0 otherwise. Each person has a probability $\frac{1}{100}$ of getting their hat and therefore

$$E(X_i) = \frac{1}{100}.$$

The total number of people who get their own hat back is just

$$X = \sum_{i=1}^{100} X_i,$$

and therefore by linearity of expected value

$$E(X) = \sum_{i=1}^{100} E(X_i) = \frac{100}{100} = 1.$$

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14. A pair of two cards are drawn successively from a deck of 52 cards. Find the probability that the second card is higher in rank than the first card. Hint: Show that $1 = P(\text{higher}) + P(\text{lower}) + P(\text{same})$ and use the fact that $P(\text{higher}) = P(\text{lower})$.

Solution: Following the hint, there are three possible mutually exclusive outcomes (both the same, second is higher, second is lower). This means that

$$1 = P(\text{higher}) + P(\text{same}) + P(\text{lower}).$$

By symmetry $P(\text{higher}) = P(\text{lower})$, and therefore

$$P(\text{higher}) = \frac{1}{2}(1 - P(\text{same})).$$

The probability they are the same rank is just the probability that the second card is a specific rank that has already been chosen

$$P(\text{same}) = \frac{3}{51}.$$

Therefore

$$P(\text{higher}) = \frac{1}{2}\left(1 - \frac{3}{51}\right) = \frac{8}{17}.$$

15. A started motor used in a space vehicle has a high rate of reliability and was reputed to start on any given occasion with probability .99999. What is the probability of at least one failure in the next 10,000 starts?

Solution: The number of failures Y follows a Binomial distribution with $p = 1 - 0.99999 = 0.00001$ and $n = 10,000$.

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{10,000}{0}(0.99999)^{10{,}000} = 1 - (0.99999)^{10{,}000}$$

This is horrible expression. Since n is large and p is very small, we can make this calculation simpler and approximate Y by a Poisson random variable X with $\lambda = np = 0.1$ giving

$$P(Y \geq 1) \approx P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.1} \approx 0.0952.$$

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15. Articles coming through an inspection line are inspected by two successive inspectors. When a defective article comes through the inspection line, the probability that it gets by the first inspector is .1. The second inspector will miss five out of ten of the defective items that get past the first inspector. What is the probability that a defective item gets by both inspectors?

Solution: Let I_1, I_2 be the events that the article gets by inspector 1 and 2 respectively. We have

$$P(I_1 \cap I_2) = P(I_2|I_1)P(I_1) = \left(\frac{1}{2}\right)\left(\frac{1}{10}\right) = \frac{1}{20}.$$

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16. Suppose a class has 150 students.

(a) A mad professor decides to pick 50 students equally likely at random and give those 50 students a free A (I would never do this). Suppose you are in the class, what is the probability that you will be one of the 50 students chosen and hence receive a free A?

(b) Exams are handed back one by one in a random order (i.e. an ordering of the 150 students is chosen equally likely at random from all possible orderings of students and the exams are handed back in that order). If Alice and Bob are two students in class, what is the probability that Alice will get her test back before Bob?

Solution:

(a) Its intuitively clear that the probability you get an A is $\frac{1}{3}$. To see this more carefully we can count. Assuming you get chosen, there are $\binom{149}{49}$ ways to choose the remaining students and $\binom{150}{50}$ ways to assign 50 students an A out of the 150. Therefore

$$P(\text{you getting an A}) = \frac{\binom{149}{49}}{\binom{150}{50}} = \frac{\frac{149!}{100!49!}}{\frac{150!}{100!50!}} = \frac{149!(50!)}{150!(49!)} = \frac{50}{150} = \frac{1}{3}.$$

As an alternative solution, suppose we have 150 slots with 50 of them designated as A slots. Students are then assigned to these slots randomly. since all assignments are equally likely, the probability of being assigned an A is just

$$P(\text{you getting an A}) = \frac{\#\text{ of A slots}}{\#\text{ of total slots}} = \frac{50}{150} = \frac{1}{3}.$$

(b) Out of all possible ways to hand back the exams exactly half of them have Alice before Bob. Therefore

$$P(\text{Alice get exam back before Bob}) = \frac{1}{2}.$$

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17. I have a bag with 3 coins in it. One of them is a fair coin, but the others are biased trick coins. When flipped, the three coins come up heads with probability 0.5, 0.6, 0.1 respectively.

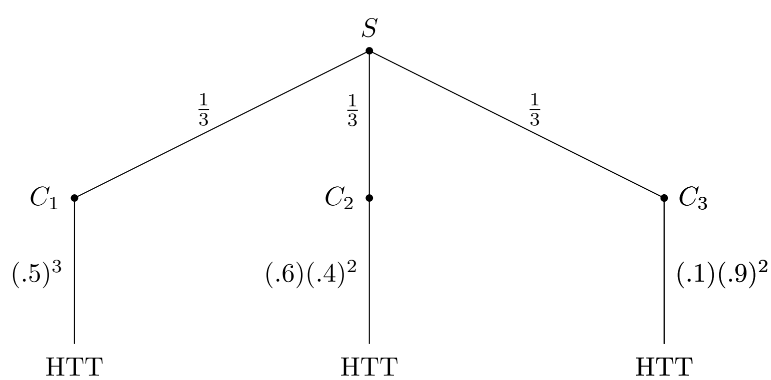
Suppose that I pick one of these three coins uniformly at random and flip it three times.

(a) What is P(HTT)? (That is, it comes up heads on the first flip and tails on the second and third flips.)

(b) Assuming that the three flips, in order, are HTT, what is the probability that the coin that I picked was the fair coin?

Solution:

(a) Let C_1, C_2, C_3 be the event that you pick the coins with probabilities $0.5, 0.6, 0.1$ respectively. We can write a probability tree for the event HTT as



Therefore by the law of total probability,

$$P(HTT) = \frac{1}{3}\left((.5)^3 + (.6)(.4)^2 + (.1)(.9)^2\right) = \frac{302}{3000}$$

(b) Using Bayes rule

$$P(C_1|HTT) = \frac{P(HTT|C_1)P(C_1)}{P(HTT)} = \frac{(.5)^3\left(\frac{1}{3}\right)}{\frac{302}{3000}} = \frac{125}{302}.$$

18. Ten teams are playing in a tournament. In the first round, the teams are randomly assigned to games 1,2,3,4,5. In how many ways can the teams be assigned to the games? How about $2n$ teams and n games?

Solution: This is just the number of ways to $2n$ teams into n games of 2 teams each. This is given by the multinomial coefficient

$$\binom{2n}{\underbrace{2\,2\,\ldots\,2}_{n\text{ times}}} = \frac{(2n)!}{2^n}.$$

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19. If $Y \sim \text{Binomial}(n, p)$, show that

$$P(Y > 1 | Y \geq 1) = \frac{1 - (1-p)^n - np(1-p)^{n-1}}{1 - (1-p)^n}$$

Solution: Note that $P(\{Y \geq 1\} \cap \{Y > 1\}) = P(Y > 1)$. By the definition of conditional probability this gives

$$\begin{aligned} P(Y > 1 | Y \geq 1) &= \frac{P(Y > 1)}{P(Y \geq 1)} = \frac{1 - P(Y \leq 1)}{1 - P(Y = 0)} \\ &= \frac{1 - \binom{n}{0}(1-p)^n - \binom{n}{1}p(1-p)^{n-1}}{1 - \binom{n}{0}(1-p)^n} \\ &= \frac{1 - (1-p)^n - np(1-p)^{n-1}}{1 - (1-p)^n} \end{aligned}$$
