Q 1.1.

Need normalization.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^1 \int_{x^2}^{1} cx \,dy\,dx$$

$$= c\int_0^1 x(1-x^2)\,dx = c\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_0^1$$

$$= \frac{c}{4} = 1$$

Therefore $c = 4$.

Q 1.2

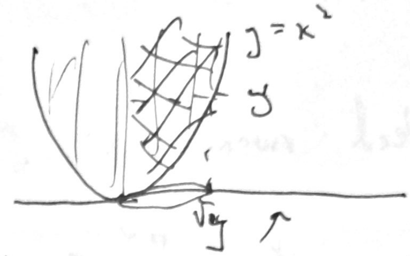
$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{1} 4x^2\,dy\,dx = \int_0^1 4x^2(1-x^2)\,dx$$

$$= \left[\frac{4}{3}x^3 - \frac{4}{5}x^5\right]_0^1 = \frac{4}{3} - \frac{4}{5} = \frac{8}{15}$$

Q 1.3

What is $f(x|y) = \dfrac{f(x,y)}{f_Y(y)}$?



since $0 \le x \le 1$, $0 \le x^2 \le y$
$\Rightarrow 0 \le x \le \sqrt{y}$

the marginal is $f_Y(y) = \int_0^{\sqrt{y}} f(x,y)\,dx$

$$= \int_0^{\sqrt{y}} 4x\,dx = 2x^2 \Big|_0^{\sqrt{y}} = 2y \quad , \quad 0 \le y \le 1$$

So $f(x|y) = \begin{cases} 2\dfrac{x}{y} & 0 \le x \le 1 ,\ 0 \le x^2 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

Q2

$X = 2Y + Z$ — Z, Y independent.

$Var(X) = 40$

$Var(Y) = 2$

Q2.1

$Var(X) = Var(2Y + Z)$ ↓ property of variance

$= Var(2Y) + Var(Z) + Cov(2Y, Z)$

$= 4Var(Y) + Var(Z)$ ‖ 0

Y, Z independent.

$Var(Z) = Var(X) - 4\,Var(Y)$

$= 40 - 4(2)$

$= 40 - 8 = 32.$

Q 2.2

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(2Y+Z, Y)$$ Bilinearity of cova

$$= \mathrm{Cov}(2Y, Y) + \mathrm{Cov}(Z, Y)$$

$$= 2\underbrace{\mathrm{Cov}(Y,Y)}_{=\mathrm{Var}(Y)} + \underbrace{\mathrm{Cov}(Z,Y)}_{=0}$$

therefore

$$\mathrm{Cov}(X,Y) = 2\,\mathrm{Var}(Y)$$

$$= 2(2) = 4.$$

Q3.1

$A, B \sim U(0,2)$ independent

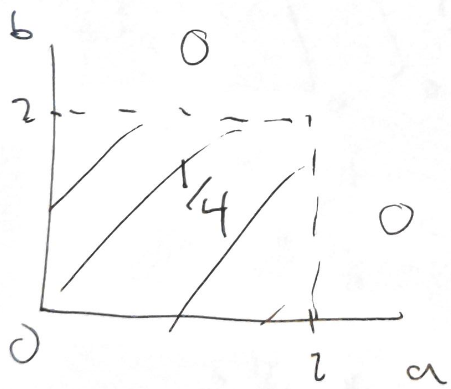
$f_A(a) = 1/2$, $f_B(b) = 1/2$ $\quad 0 \le a, b \le 2.$

$f(a,b) = f_A(a) f_B(b) = \frac{1}{2}\left(\frac{1}{2}\right) = 1/4 \quad 0 \le a, b \le 2.$

↑ independence

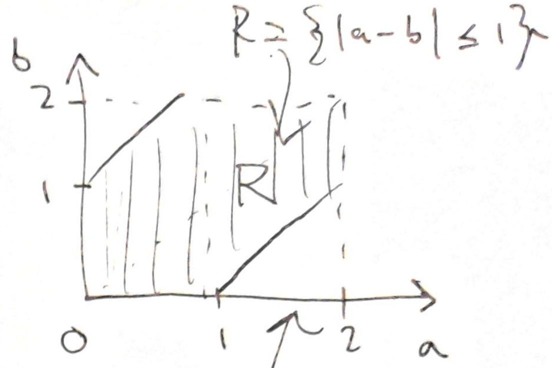
therefore

$$f(a,b) = \begin{cases} 1/4 & 0 \le a, b \le 2 \\ 0 & \text{otherwise} \end{cases}$$



Q3.2 $\boxed{\text{Method 1}}$

$$P(|A-B| \le 1)$$



$$= \iint_R \tfrac{1}{4}\, da\, db.$$

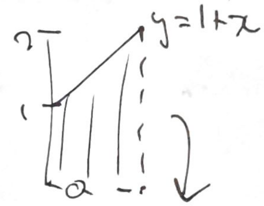
Area of triangle

Area of square

$$= \tfrac{1}{4}\,\text{Area}(R) = \tfrac{1}{4}\left(2(2) - 2\left(\tfrac{1}{2}\right)\right)$$

$$= 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

$\boxed{\text{Method 2.}}$



$$P(|A-B| \le 1) = \int_0^1 \int_0^{1+x} \tfrac{1}{4}\, dy\, dx + \int_1^2 \int_{x-1}^{2} \tfrac{1}{4}\, dy\, dx$$

$$= \tfrac{1}{4}\int_0^1 1+x\, dx + \tfrac{1}{4}\int_1^2 2-(x-1)\, dx$$

$$= \tfrac{1}{4}\left[x + \tfrac{1}{2}x^2\right]_0^1 + \tfrac{1}{4}\left[3x - \tfrac{1}{2}x^2\right]_1^2$$

$$= \tfrac{1}{4}\left(\tfrac{3}{2}\right) + \tfrac{1}{4}\left[(6-2) - \left(3-\tfrac{1}{2}\right)\right]$$

$$= \frac{3}{8} + \tfrac{3}{8} = \cancel{\tfrac{4}{8}} = \tfrac{3}{4}$$

Q4.1

$X_1 \sim U(0,1)$ - Break 1.

Given $X_1 = x_1$ $X_2 \sim U(0, x_1)$ - Break 2.

So
$$\begin{cases} f_{X_1}(x_1) = 1 & 0 \le x_1 \le 1 \\ f(x_2 | x_1) = \frac{1}{x_1}, & 0 \le x_2 \le x_1 \end{cases}$$

The joint pdf is then

$$f(x_1, x_2) = f(x_2 | x_1) f_{X_1}(x_1)$$

$$= \begin{cases} 1/x_1, & 0 \le x_1 \le 1, \ 0 \le x_2 \le x_1 \\ 0 & \text{otherwise} \end{cases}$$

Q 4.2 Method 1

$$E(X_1 X_2) = E(E(X_1 X_2 | X_1))$$

$$= E(X_1 E(X_2 | X_1))$$

Since $E(X_2 | \cancel{X_1}) = \int_0^{x_1} \frac{x_2}{x_1} \, dx_2 = \frac{1}{2}\frac{x_2^2}{x_1}\Big|_0^{x_1} = \frac{1}{2} x_1$

Then

$$E(X_1 E(X_2 | X_1)) = E(\tfrac{1}{2} X_1^2) = \tfrac{1}{2} E(X_1^2)$$

$$= \tfrac{1}{2} \int_0^1 x_1^2 \, dx_1$$

$$= \tfrac{1}{2} \left(\tfrac{1}{3} x_1^3 \Big|_0^1 \right) = \tfrac{1}{6}.$$

Method 2 (Almost the same)

$$E(X_1 X_2) = \int_0^1 \int_0^{x_1} \cancel{\frac{x_1}{x_1}} x_2 \, dx_2 \, dx_1$$

$$= \int_0^1 \tfrac{1}{2} x_2^2 \Big|_{x_2=0}^{x_2=x_1} dx_1 = \int_0^1 \tfrac{1}{2} x_1^2 \, dx_1 = \tfrac{1}{2}\left(\tfrac{1}{3} x_1^3 \Big|_0^1\right)$$

$$= \tfrac{1}{6}.$$