Midterm 2 Practice Problems

These are a collection of practice problems for the second midterm exam. If you can do these problems (without looking at solutions), there is a high probability that you will do well on the exam.

1.

(a) Suppose that X is uniform on $[0,1]$. Compute the pdf and cdf of X.

(b) If $Y = 2X + 5$, compute the pdf and cdf of Y.

2.

(a) Suppose that X has probability density function $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Compute the cdf, $F_X(x)$.

(b) If $Y = X^2$, compute the pdf and cdf of Y.

3. Let R be the rate at which customers are served in a queue. Suppose that R is exponential with pdf $f(r) = 2e^{-2r}$ on $[0, \infty)$. Find the pdf of the waiting time per customer $T = 1/R$.

4. A continuous random variable X has pdf $f(x) = x + ax^2$ on $[0,1]$. Find a, the cdf and $P(.5 < X < 1)$.

5. Suppose X has range $[0,1]$ and has cdf

$$F(x) = \begin{cases} x^2 & 0 \leq x \leq 1. \\ 0 & \text{otherwise.} \end{cases}$$

Compute $P(1/2 < X < 3/4)$.

6. Let X be a random variable with range $[0,1]$ and cdf

$$F(x) = \begin{cases} 2x^2 - x^4 & 0 \leq x \leq 1. \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $P(1/4 < X < 3/4)$.

(b) What is the pdf of X?

7. Suppose a random circle is formed by randomly choosing a radius uniformly in $[0,1]$.

(a) What is the expected area of the circle?

(b) What is the variance of the area of the circle?

8. Suppose that buses arrive are scheduled to arrive at a bus stop at noon but are always X minutes late, where X is an exponential random variable with probability density function $f_X(x) = \lambda e^{-\lambda x}$. Suppose that you arrive at the bus stop precisely at noon.

(a) Compute the probability that you have to wait for more than five minutes for the bus to arrive.

(b) Suppose that you have already waiting for 10 minutes. Compute the probability that you have to wait an additional five minutes or more.

9. A bus arrives every 15 minutes starting at $7:00$am. You walk to the station and arrive uniformly at a random time between $7:10$am and $7:30$am. What is the pdf for the amount of time spent waiting for the bus?

10. Let $\phi(z)$ and $\Phi(z)$ be the pdf and cdf of the standard normal distribution. Suppose that Z is a standard normal random variable and let $X = 3Z + 1$.

(a) Express $F_X(x) = P(X \leq x)$ in terms of Φ.

(b) Use this to find the pdf of X in terms of ϕ.

(c) Find $P(-1 \leq X \leq 1)$.

(d) Recall that the probability that Z is within one standard deviation of its mean is approximately 68%. What is the probability that X is within one standard deviation of its mean?

11. Suppose $Z \sim N(0, 1)$ and $Y = X^2$. What is the pdf of Y?

12. Let $X_1, X_2, \ldots X_n$ be iid standard normal random variables. Let

$$Y_n = X_1^2 + X_2^2 + \ldots + X_n^2.$$

(a) Show that $E(X_j^2) = 1$.

(b) Set up an integral for $E(X_j^4)$. For extra credit, use integration by parts to show that

$$E(X_j^4) = 3.$$

(If you can't figure out how to do this, just use this number in the next part)

(c) Deduce from (a) and (b) that $\text{Var}(X_j^2) = 2$.

(d) Use the Central Limit Theorem to approximate $P(Y_{100} > 110)$.

13. Suppose $X_1, X_2, \dots X_{100}$ are iid with mean $\mu = E(X_j) = 1/5$ and variance $\sigma^2 = \text{Var}(X_j) = 1/9$. Use the central limit theorem to estimate

$$P\left(\sum_{j=1}^{100} X_j < 30\right).$$

14. Let $X \sim \text{Binomial}(100, 1/3)$. Use the Central Limit Theorem to give an approximation of $P(X \leq 30)$.

15. The average IQ in a population is 100 with standard deviation 15 (by definition IQ is actually curved so that this is case). What is the probability that a randomly selected group of 100 people has an average IQ above 115?

16. Let X and Y be independent normal random variables, where $X \sim N(2, 5)$ and $Y \sim N(5, 9)$, let $W = 3X - 2Y + 1$.

(a) Compute $E(W)$ and $\text{Var}(W)$.

(b) It is known that the sum of independent normal random variables is normal (take this as a given). Compute $P(W \leq 6)$.

17. Let $X \sim U(a, b)$. Compute $E(X)$ and $\text{Var}(X)$.

18. Compute the median for an exponential distribution with rate λ.

19. Let X and Y be independent Bernoulli(.5) random variables. Define

$$S = X + Y \quad \text{and} \quad T = X - Y.$$

(a) Find the joint and marginal distributions for S and T.

(b) Are S and T independent?

20. Data is taken on the height and shoe size of a sample of Brown students. Height is coded by 3 values 1 (short), 2 (medium), 3 (tall) and similarly for shoe size 1 (small), 2 (medium), 3 (large). The total counts are displayed in the following table

Shoe\ Height	1	2	3
1	234	225	84
2	180	453	161
3	39	192	157

Let X be the coded shoe size and Y be the coded shoe height of a random person in the sample.

(a) Find the joint and marginal distributions of X and Y.

(b) Are X and Y independent?

21. Let X and Y be two continuous random variables with joint pdf given by:

$$f(x,y) = \begin{cases} c(x^2y) & 0 \le x \le y,\ x+y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid density function.

(b) What is the expectation of X?

(c) What is the marginal distribution for Y?

22. Let X_1 and X_2 be two continuous random variables with joint pdf given by:

$$f(x_1,x_2) = \begin{cases} cx_1x_2 & 0 \le x_1 \le 1,\ 0 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid density function.

(b) Find the marginal densities for X_1 and X_2.

(c) Are X_1 and X_2 independent?

(d) What is the probability that $X_1 < 1/4$ given that $X_2 = 1/2$?

(e) Find the expectation of X_1X_2.

23. Let X and Y be two continuous random variables with joint pdf given by:

$$f(x,y) = \begin{cases} cx^2y(1+y) & 0 \le x \le 3,\ 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid density function.

(b) Find $P(1 \le X \le 2, 0 \le Y \le 1)$.

(c) Determine the joint cdf $F(x,y)$ of X and Y.

(d) Find the marginal cdf $F_X(x)$.

(e) Find the marginal pdf $f_X(x)$ directly from $f(x,y)$ and check that it is the derivative of $F_X(x)$.

(f) Are X and Y independent?

24. Suppose Y_1 and Y_2 iid random variables sampled from a Uniform$(0, 1)$ distribution.

(a) What is the joint density function $f(x_1, x_2)$?

(b) What is the probability that $X_1 + X_2$ is less than $\frac{1}{3}$?

(c) Find the expectation of $X_1 + X_2$.

25. Let Y_1 denote the weight (in tons) of a bulk item stocked by a supplier at the beginning of a week and suppose that Y_1 has a uniform distribution over the interval $0 \le y_1 \le 1$. Let Y_2 denote the amount (by weight) of this item sold by the supplier during the week and suppose that Y_2 has as uniform distribution over the interval $0 \le y_2 \le y_1$, where y_1 is specific value of Y_1.

(a) Find the joint pdf for Y_1 and Y_2.

(b) If the supplier stocks a half-ton of the item, what is the probability that she sells more than a quarter-ton?

(c) If it is known that the supplier sold a quarter-ton of the item, what is the probability that she had stocked more than a half-ton.

26. If Y_1 is uniformly distributed on the interval $(0, 1)$ and, for $0 < y_1 < 1$,

$$f(y_2|y_1) = \begin{cases} 1/y_1 & 0 \le y_2 \le y_1, \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the joint pdf of Y_1 and Y_2.

(b) Find the marginal pdf for Y_2.

27. Let X and Y be two independent exponential random variables with parameters λ_X and λ_Y. Let $Z = X + Y$. Find the covariance of X and Z.

28. Let X and Y be two random variables and let r, s, t, u be real numbers.

(a) Show that $\text{Cov}(X + s, Y + u) = \text{Cov}(X, Y)$.

(b) Show that $\text{Cov}(rX, tY) = rt\text{Cov}(X, Y)$.

(c) Show that $\text{Cov}(rX + s, tY + u) = rt\text{Cov}(X, Y)$.

29. Let X and Y be two discrete random variables whose joint and marginal distributions are partially given in the following table.

$X\backslash Y$	1	2	3	$p_X(x)$
1	1/6	0	·	1/3
2	·	1/4	·	1/3
3	·	·	1/4	·
$p_Y(y)$	1/6	1/3	·	1

(a) Complete the table.

(b) Are X and Y independent?

30. Let X and Y be two discrete random variables which take values $\{1, 2, 3, 4\}$. The following formula gives their joint distribution,

$$p(i, j) = P(X = i, Y = j) = \frac{i + j}{80},$$

Compute each of the following:

(a) $P(X = Y)$.

(b) $P(XY = 6)$.

(c) $P(1 \leq X \leq 2, 2 < Y \leq 4)$.

31. Toss a fair coin three times. Let X = the number of heads on the first toss, Y = the total number of heads on the last two tosses, and Z = the number of heads on the first two tosses.

(a) Give the joint probability table for X and Y. Compute $\text{Cov}(X, Y)$.

(b) Give the joint probability table for X and Z. Compute $\text{Cov}(X, Z)$.