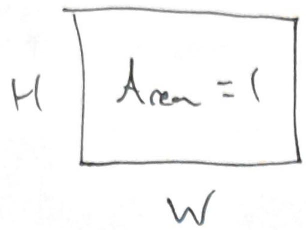
Q 2.1



$W = 1/H$

$$f_H(x) = \begin{cases} 1 & x \in \{1,2) \\ 0 & \text{otherwise} \end{cases}$$

uniform

$$E(W) = \int_1^2 1/x \, dx = \left[\ln x\right]_1^2 = \ln(2) - \overset{=0}{\ln(0)}$$
$$= \underline{\ln(2)}$$

Also

$$E(W^2) = \int_1^2 1/x^2 \, dx = \left[-1/x\right]_1^2 = 1 - 1/2 = \underline{1/2}$$

Therefore

$$Var(W) = E(W^2) - (E(W))^2$$
$$= 1/2 - (\ln(2))^2$$

Q 2.2

the distribution for $W = 1/H$ is

$$F_W(\omega) = P(W \le \omega) = P(1/H \le \omega)$$

$$= P(1/\omega \le H)$$

$$= 1 - F_H(1/\omega)$$

Since $F_H(x) = \begin{cases} x-1, & x \in [1,2] \\ 0 & x \le 1 \\ 1 & x \ge 2. \end{cases}$

Taking the derivative

$$f_W(\omega) = -\frac{d}{d\omega} F_H(1/\omega) = f_H(1/\omega)\, 1/\omega^2$$

chain rule

$$= \begin{cases} 1/\omega^2 & 1 \le 1/\omega \le 2. \\ 0 & \text{otherwise} \end{cases}$$

or

$$= \begin{cases} 1/\omega^2 & 1/2 \le \omega \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

Q 3.1

$$X_i \sim Exp(1/6) \Rightarrow EX_i = 6 \,, \quad Var(X_i) = 36$$

$$I = \frac{1}{10} \sum_{i=1}^{100} X_i$$

So $E(I) = \frac{1}{10} \sum_{i=1}^{100} E(X_i) = \frac{100}{10} 6 = \underline{\underline{60.}}$

and

$$Var(I) = \frac{1}{100} Var\left(\sum_{i=1}^{100} X_i\right) \overset{\text{independence}}{=} \frac{100}{100} Var(X_i)$$

$$= \underline{\underline{36.}}$$

$\underline{\$60} > \$57$ Yes

average earned per tree

cost per tree.

Q 3.2

$$P(I < 57) = ?$$

Standardize

$$Z = \frac{I - \overset{E(I)}{60}}{\underset{\sqrt{Var(I)}}{\sqrt{36}}} \approx N(0,1) \quad \text{by CLT}$$

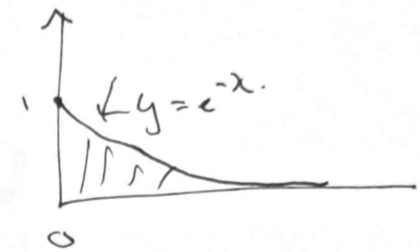
Therefore

$$P(I < 57) = P\left(Z < \frac{57-60}{\sqrt{36}} = -\tfrac{1}{2}\right)$$

$$\approx \Phi(-\tfrac{1}{2}) \approx .3085$$

31% chance of losing money.

Q 4.1



Normalization

$$\int_0^\infty \int_0^{e^{-x}} cy \, dy \, dx = \int_0^\infty c \, \tfrac{1}{2} y^2 \Big|_0^{e^{-x}} dx$$

$$= \int_0^\infty \frac{c}{2} e^{-2x} \, dx$$

$$= -\frac{c}{4} e^{-2x} \Big|_0^\infty = \frac{c}{4} = 1$$

Therefore $c = 4$.

Q 4.2

$$E(e^X) = \int_0^\infty \int_0^{e^{-x}} e^x 4y \, dy \, dx$$

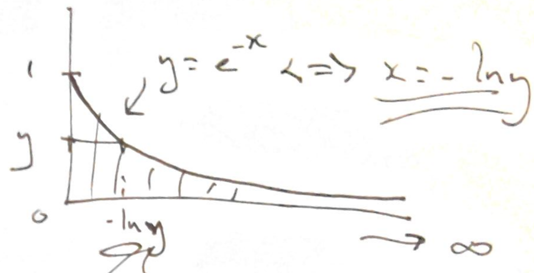
$$= \int_0^\infty 4e^x \tfrac{1}{2} y^2 \Big|_0^{e^{-x}} dx = \int_0^\infty 2 e^x e^{-2x} \, dx$$

$$= \int_0^\infty 2 e^{-x} \, dx = 2.$$

Q 4.3

Conditional density given by

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$



The marginal is

$$f_Y(y) = \int_0^{-\ln y} 4y \, dx = -4y \ln y \qquad 0 \le y \le 1$$

Therefore since $\frac{4y}{-4y\ln y} = \frac{1}{-\ln y}$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} -\frac{1}{\ln y}, & 0 \le x \le -\ln y \\ 0 & \text{otherwise.} \end{cases}$$

for each $0 \le y \le 1$

Q5

X, Y independent. $Var(X) = 2$, $Var(Y) = 8$.

$Z = 3X + Y$.

$$Cov(X, Z) = cov(X, 3X + Y) = \underbrace{Cov(X, 3X)}_{= 3Var(X)} + \underbrace{Cov(X, Y)}_{= 0}$$

$$= 3\underbrace{Var(X)}_{= 2} + 0.$$

$$= 6$$

$$Cov(Y, Z) = Cov(Y, 3X + Y) = \underbrace{Cov(Y, 3X)}_{= 0} + \underbrace{Cov(Y, Y)}_{= Var(Y).}$$

$$= 8.$$