Q2 Let

E_i - Keys in place i

F_i - Find keys in place i

We are given

$$P(F_i \mid E_i) = p_i \quad , \quad P(E_i) = 1/4$$

2.1 Want $P(E_3 \mid \bar{F}_3)$

Use Bayes

$$P(E_3 \mid \bar{F}_3) = \frac{\overbrace{P(\bar{F}_3 \mid E_3)}^{1-p_3} \, \overbrace{P(E_3)}^{1/4}}{\sum_i P(\bar{F}_3 \mid E_i) P(E_i)}$$

Note

$$\sum_i P(\bar{F}_3 \mid E_i) P(E_i) = P(E_1) + P(E_2) + \frac{(1-p_3)}{4} + P(E_4)$$

$$= \frac{4 - p_3}{4}$$

Therefore

$$P(E_3 \mid \bar{F}_3) = \frac{\frac{(1-p_3)}{\cancel{4}}}{\frac{4-p_3}{\cancel{4}}} = \frac{(1-p_3)}{4-p_3}$$

2.2 Want to calculate

$$P(E_1 \cup E_2 | \bar{F}_3) = P(E_1 | \bar{F}_3) + P(E_2 | \bar{F}_3)$$

Use Bayes for $i = 1, 4$.

$$P(E_i | \bar{F}_3) = \frac{\overbrace{P(\bar{F}_3 | E_i)}^{=1}\,\overbrace{P(E_i)}^{=1/4}}{\underbrace{P(\bar{F}_3)}_{=\frac{4-p_3}{4}}}$$

from last problem

$$= \frac{1}{4-p_3}$$

$\boxed{\text{SANITY CHECK.}}$

$$\sum_i P(E_i | \bar{F}_3) = \frac{3}{4-p_3} + \frac{(1-p_3)}{4-p_3} = \frac{4-p_3}{4-p_3} = 1 \checkmark$$

Q 3

$$f(x,y) = \begin{cases} c/x & 0 < x \le 1,\ 0 \le y \le 2x \le 2. \\ 0 & \text{otherwise} \end{cases}$$

3.1

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1 \left[\int_0^{2x} c/x \; dy\right] dx$$

$$= \int_0^1 2c \; dx = 2c = 1$$

$$\Rightarrow \quad c = 1/2.$$

3.2

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{2x} \frac{1}{2x}\,dy = 1$$

$$= \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{y/2}^{1} 1/2x \; dx = -1/2 \ln(y/2) & \text{if } \underline{0 \le y \le 2} \\ 0 & \text{otherwise} \end{cases}$$

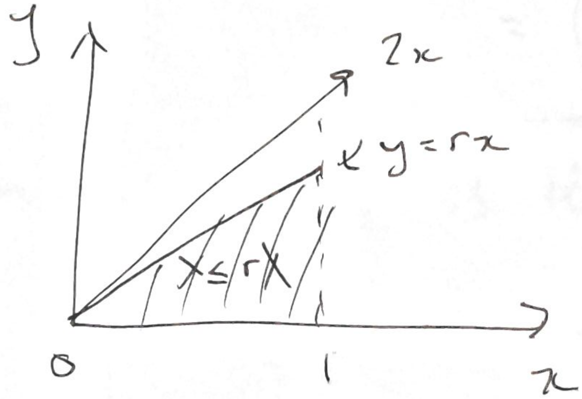
3.3

Since $0 \le Y \le 2X$, $0 \le R = Y/X \le 2$

the CDF is: if $r \in (0,2)$

$$F_R(r) = P(R \le r) = P(Y \le rX) = \int_0^1 \int_0^{rx} \frac{1}{2x}\, dy\, dx$$

$$= \int_0^1 \frac{r}{2}\, dx = r/2.$$



Therefore $f_R(r) = \begin{cases} 1/2 & r \in (0,2) \\ 0 & \text{otherwise.} \end{cases}$

$\sim \mathcal{U}(0,2)$.

Q4

4.1 For a given TA, each student has prob $p = 1/5$ of going to those office hours. Therefore

# of students at a given TAs office hours

$$\sim \text{Binomial}(100, 1/5)$$

therefore

$$P(10 \text{ students}) = \binom{100}{10} (1/5)^{10} (1 - 1/5)^{100-10}$$

$$= \underline{\underline{\binom{100}{10} 4^{90} 5^{-100}}}$$

4.2 Let

$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ TA has 10 students} \\ 0 & \text{otherwise} \end{cases}$$

Note: X_i are NOT independent.

$$X = \sum_{i=1}^{5} X_i = \# \text{ of TAs who got exactly 10 students}$$

By linearity of Expectation

$$E(X) = \sum_{i=1}^{5} E(X_i) = \underline{\underline{5 \binom{100}{10} 4^{90} 5^{-100}}}$$

Q5

$$f(x|\theta) = \begin{cases} \frac{4x^3}{\theta^4} e^{-x^4/\theta^4} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

5.1 The Likelihood function is

$$L(x_1, x_2, \dots, x_n | \theta) = \prod_{i=1}^{n} f(x_i | \theta)$$

$$= \prod_{i=1}^{n} \left[\frac{4x_i^3}{\theta^4} e^{-x_i^4/\theta^4} \right] = \left[\prod_{i=1}^{n} 4x_i^3 \right] \theta^{-4n} e^{-\left(\sum_{i=1}^{n} x_i^4\right)/\theta^4}$$

$$= h(x_1, \dots, x_n) \, g\left(\sum_{i=1}^{n} x_i^4, \theta\right).$$

Therefore by the factorization method

$u = \sum_{i=1}^{n} X_i^4$ is a sufficient statistic.

5.2 (by part 1)

$$\ln(L(\theta)) = \ln(h(x_1,\dots,x_n)) + \ln\left(g\left(\sum_{i=1}^{n} x_i^4, \theta\right)\right)$$

$$= \ln(h) - 4n\ln(\theta) - \theta^{-4}\left(\sum_{i=1}^{n} x_i^4\right)$$

Take the derivative

$$\frac{\partial}{\partial\theta}\ln(L(\theta)) = -\frac{4n}{\theta} + \frac{4\left(\sum_{i=1}^{n} x_i^4\right)}{\theta^5} = 0$$

$$\Rightarrow \quad \theta^4 = 1/n \sum_{i=1}^{n} x_i^4$$

So

$$\hat{\theta}_{MLE} = \left(1/n \sum_{i=1}^{n} X_i^4\right)^{1/4}$$

5.3

Consider

$$F_{X_i^4}(x) = P(X_i^4 \le x) = P(X_i \le x^{1/4})$$

$$= F_{X_i}(x^{1/4})$$

Taking the derivative, if $x \geq 0$,

$$f_{X_i^4}(x) = \left(f_{X_i}(x^{1/4}) \right) 1/4 \, x^{-3/4}$$

$$= \left(\frac{\cancel{4x^{3/4}}}{\theta^4} e^{-x/\theta^4} \right) \frac{1}{\cancel{4x^{3/4}}} =$$

$$= \begin{cases} 1/\theta^4 \, e^{-x/\theta^4} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

therefore $X_i^4 \sim \text{Exponential}(1/\theta^4)$

with mean $\mathbb{E} X_i^4 = \theta^4$.

5.4

Since X_i^4 is exponential it has finite variance.

By the WLLN

$$\frac{1}{n}\sum_{i=1}^{n} X_i^4 \xrightarrow{P} \theta^4$$

By properties of convergence in probability,

$$\hat{\theta}_{MLE} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^4\right)^{1/4} \xrightarrow{P} \left(\theta^4\right)^{1/4} = \theta.$$

Therefore $\hat{\theta}_{MLE}$ is consistent.

6) We want a one-sided confidence interval on the mean μ, (θ_L, ∞).

Setting $\theta_L = \bar{X} - t \frac{s}{\sqrt{4}}$

We want
$$P(\theta_L < \mu) = 1 - \alpha.$$

$$\Rightarrow P\left(\frac{\bar{X} - \mu}{s/\sqrt{4}} < t\right) = 1 - \alpha$$

Note $T = \frac{\bar{X} - \mu}{s/\sqrt{4}} \sim t$ distribution with dof $= 4 - 1 = 3$.

- since the sample size $n = 4 < 30$
- since population is normal.

Using t-table $P(T < \overset{4.541}{\overset{\shortparallel}{t_{.01,3}}}) = .99$.

Therefore $\theta_L = 160 - \frac{4.541(4)}{2} = 160 - 4.541(2)$
> 150

Therefore you should serve the burgers.