Central Limit Theorem

Law of large numbers.

$X_1, X_2, X_3, \ldots X_n$ independent and indentically distributed. iid

$\mu = E(X_i)$

$\sigma^2 = Var(X_i)$

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$\mathbb{E}(\overline{X}_n) = \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i)$$

$$= \frac{\cancel{n}}{\cancel{n}} \mu = \mu$$

$$Var(\overline{X}_n) = Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

independent

$$= \frac{1}{n^2} \sum_{i=1}^{n} \overset{\sigma^2}{\overset{\shortparallel}{Var(X_i)}} = \frac{\cancel{n}}{n^{\cancel{2}}} \sigma^2$$

$$= \frac{\sigma^2}{n}$$

Summarize

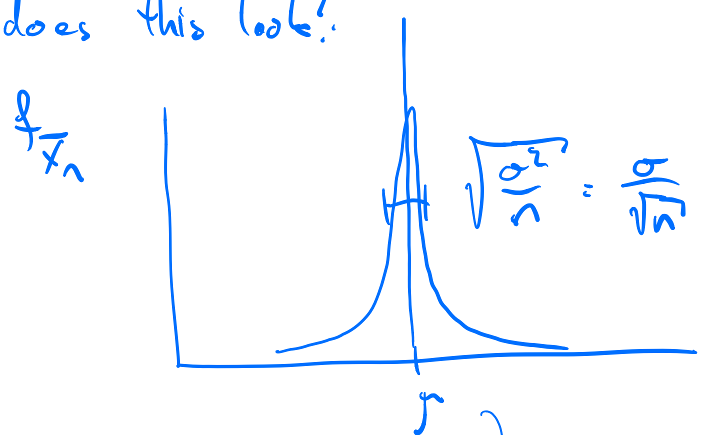
$$\mathbb{E}(\bar{X}_n) = \mu \quad , \quad \mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$$

By Chebyshev

$$P(|\bar{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{\varepsilon n} \longrightarrow 0$$

as $n \to \infty$.

How does this look?



↳ Law of large numbers.

What is the shape of $f_{\bar{X}_n}$ (the distribution of $\bar{X}_n$)?

Lets standardize $\bar{X}_n$.

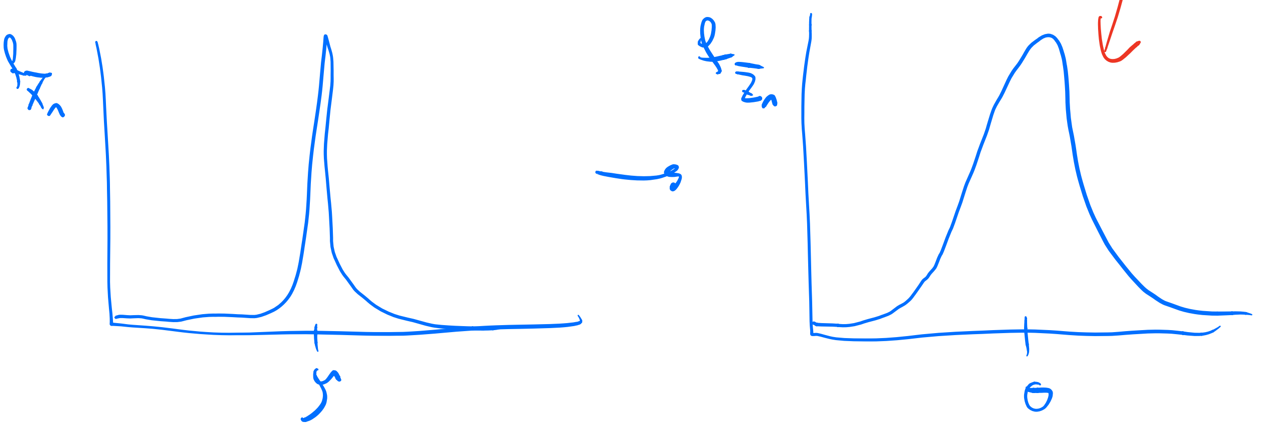
Define

$$\bar{Z}_n = \frac{\bar{X}_n - \mathbb{E}(\bar{X}_n)}{\sqrt{\mathrm{Var}(\bar{X}_n)}} = \frac{\bar{X}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}}$$

$$= \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}}$$

$$\mathbb{E}\bar{Z}_n = 0 \quad , \quad \mathrm{Var}(\bar{Z}_n) = 1 \, .$$

approaches $N(0,1)$.



What happens to $\bar{Z}_n$ as $n \to \infty$?

Central Limit Theorem

$$\bar{Z}_n \xrightarrow[n \to \infty]{} N(0,1)$$

In distribution

$$F_{Z_n}(z) = P(Z_n \le z) \xrightarrow[n\to\infty]{} \Phi(z) = P(Z \le z).$$

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \simeq Z \sim N(0,1)$$

this is the same as

$$\bar{X}_n \approx \frac{\sigma}{\sqrt{n}} Z + \mu \sim N(\mu, \sigma/\sqrt{n})$$

Can be used to approximate Binomial random variable

$$S_n = \sum_{i=1}^{n} X_i \quad , \quad X_i \sim \text{Bernoulli}(p)$$

independent.

$S_n \sim \text{Binomial}(n, p).$

Central limit theorem for Binomial.

$$\frac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}} \approx N(0,1)$$

$$\mathbb{E}X_i = p \quad , \quad Var(X_i) = p(1-p)$$

We find

$$\frac{\frac{S_n}{n} - p}{\sqrt{p(1-p)}/\sqrt{n}} \approx N(0,1) = Z.$$

$$S_n \approx np + \sqrt{np(1-p)}\, Z.$$

$$\approx N(np, np(1-p)).$$

Binomial, $\approx$ Normal.