T - Distributions

How to estimate μ when you don't know σ^2.

We have $X_1, X_2, \ldots, X_n$ random sample from an $N(\theta, \sigma^2)$. we don't know σ^2.

Want to estimate θ.

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}.$$

Pivotal Quantity.

$$Z = \frac{\bar{X} - \theta}{\sigma / \sqrt{n}} \sim N(0,1)$$

??

Gives a $(1-\alpha)$ confidence interval

$$\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Lets just estimate σ^2, by the sample Variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Random quantity.

unbiased estimator for σ^2.

Consider the new pivotal quantity

$$\bar{Z} = \frac{\bar{X} - \theta}{S/\sqrt{n}}$$

Not normal!

How is $\bar{Z}$ distributed?!

How is S^2 distributed?

χ^2 - Distribution (n-1 degrees of freedom).

$X \sim \chi^2(\nu)$

↑ # of degrees of freedom

if the pdf is

$$f_X(x) = \frac{1}{2^{\nu/2}\,\Gamma(\nu/2)}\, x^{\nu/2 - 1} e^{-x/2}, \quad x \geq 0.$$

↑ Gamma function
generalization of the
factorial.

- An example of $\mathrm{Gamma}(\nu/2, 2)$.

  Gamma distribution

__Theorem__ if $Z_1, Z_2, \ldots, Z_n$ are iid $N(0,1)$ RVs then

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

$$\sim \chi^2(n)$$ → n-degrees of

freedom

Properties if $X \sim \chi^2(r)$.

① $\mathbb{E} X = r$

② $Var(X) = 2r$.

Why is this useful?

Theorem $X_1, X_2, \ldots, X_n$ be a random sample of $N(\theta, \sigma^2)$. Then

$$Y = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2$$

① $Y \sim \chi^2(\underline{\underline{n-1}})$.

② S^2 and $\bar{X}$ are independent.

Will not prove. – A little complicated.

t - distribution. (students t-distribution).

Def Let $Z \sim N(0,1)$, $U \sim \chi^2(n)$

and Z, U are independent.

$$T = \frac{Z}{\sqrt{U/n}} \sim T(n).$$

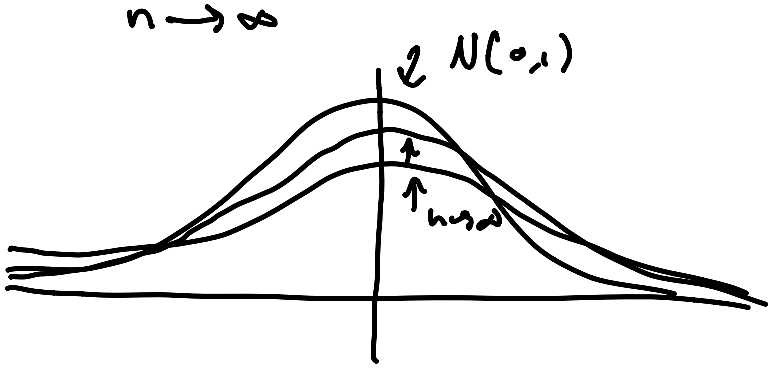
t-distribution with n degrees of freedom

Properties

① $T(n)$ has a bell shaped

- more spread out than $N(0,1)$.

② $T(n) \xrightarrow[n\to\infty]{} N(0,1)$ & CLT.

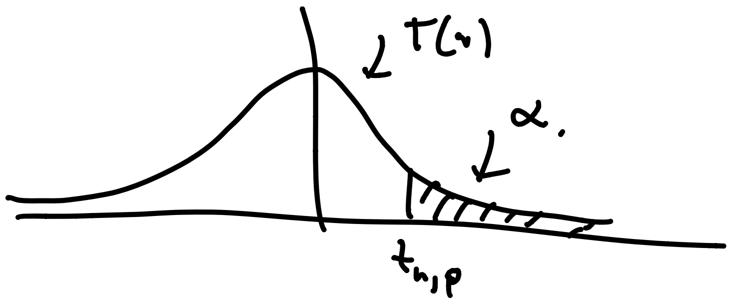


③ for $\alpha \in (0,1)$. define

$$t_{\alpha,n} = F_{T(n)}^{-1}(1-\alpha).$$

$$\Updownarrow$$

$$P(T > t_{\alpha,n}) = \alpha.$$



<u>Theorem</u> $X_1, X_2, \dots, X_n$ be a random sample of $N(\theta, \sigma^2)$, then

$$T = \frac{\bar{X} - \theta}{S/\sqrt{n}}$$

↳ replaced σ by S.

has a $T(n-1)$ <u>distribution</u>.

Specifically.

We can find a confidence interval for θ by

$$P(|\bar{T}| < t) = 1-\alpha$$

$$\Downarrow$$

$$P(|\bar{T}| \le t_{\alpha/2,n}) = 1-\alpha.$$

$$|\bar{T}| \le t_{\alpha/2,n}$$

$$\Updownarrow$$

$$-t_{\alpha/2,n} \le \frac{\bar{X}-\theta}{S/\sqrt{n}} \le t_{\alpha/2,n}$$

$$\Updownarrow$$

$$\bar{X} - t_{\alpha/2,n} S/\sqrt{n} \le \theta \le \bar{X} + t_{\alpha/2,n} S/\sqrt{n}$$

therefore

$$\left[\bar{X} - t_{\alpha/2,n} S/\sqrt{n}, \ \bar{X} + t_{\alpha/2,n} S/\sqrt{n}\right]$$

is a $1-\alpha$ confidence interval for θ.