What is the pdf and cdf of an RV defined in terms of another RV?

Distribution Method.

Ex $X = aZ + b$

$$\mathbb{E}(X) = a\mathbb{E}(Z) + b$$

$$Var(X) = a^2 Var(Z).$$

What is the distribution?

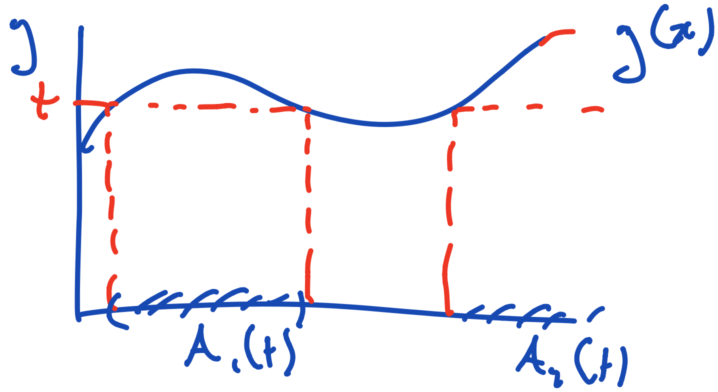
Distribution Method.

Suppose X continuous RV with $f_X(x)$ as its pdf.

Given some function $g(x)$.

What is the distribution of $Y = g(X)$.

Take.



$$\{x : g(x) < t\}$$

$$= A_1 \cup A_2$$

↑ disjoint

$$P(g(X) < t) = P(X \in A_1(t)) + P(X \in A_2(t))$$

↳ CDF of Y.

↳ Can be written in terms of the CDF of X.

- Take the derivation of both sides

$$f_Y(t) = \text{something}.$$

Example $X \sim U(0,2)$.

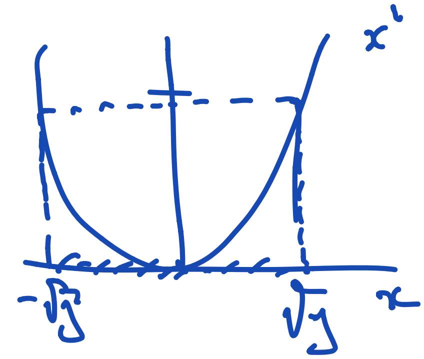
$$f_X(x) = \begin{cases} 1/2 & \text{on } 0 \le x \le 2. \\ 0 & \text{otherwise} \end{cases}$$

$\Updownarrow$

$$F_X(x) = \tfrac{1}{2}x. \quad \text{on } 0 \le x \le 2.$$

What is the pdf of $Y = X^2$?

$$F_Y(y) = P(Y \le y)$$



$$= P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - \underbrace{F_X(-\sqrt{y})}_{=0}$$

because $X \geq 0$

$$= F_X(\sqrt{y})$$

Use chain rule

$$F_Y(y) = F_X(\sqrt{y}).$$

↳ take derivative in y.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\sqrt{y}).$$

$$= F_X'(\sqrt{y}) \frac{d}{dy}\sqrt{y} = f_X(\sqrt{y})\, \frac{1}{2} \frac{1}{\sqrt{y}}.$$

$$f_Y(y) = \begin{cases} \frac{1}{4}\frac{1}{\sqrt{y}} & , 0 \le \sqrt{y} \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{4\sqrt{y}} & , 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

<u>Ex</u> $X \sim \text{Exponential}(a)$, $f_X(x) = ae^{-ax}$ on $[0,\infty)$.

What is the density of $Y = X^2$

$$F_Y(y) = P(X^2 \le y).$$

$$= P(X \le \sqrt{y})$$

$$= F_X(\sqrt{y})$$

$$f_Y(y) = f_X(\sqrt{y}) \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \frac{ae^{-a\sqrt{y}}}{2\sqrt{y}}, \quad 0 \le y < \infty.$$

Can be seen as change of variables.

$x \geq 0$.

$$f_X(x)\,dx \longleftrightarrow f_Y(y)\,dy \,.$$

$y = x^2$, $\Rightarrow$ $dy = 2x\,dx$

$$\Rightarrow dx = \frac{dy}{2\sqrt{y}}$$

$$f_X(x)\,dx = f_X(\sqrt{y})\,\frac{dy}{2\sqrt{y}} = f_Y(y)\,dy.$$

$$\text{"} f_X(x)\,dx = f_X(y)\,dy \text{"}$$

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}}$$

$\square$