Point Estimators.

Goal - to estimate some property or parameter of a distribution
known or unknown.

Estimator A parameter θ to be estimated.

Ex

- $\theta = \mu$ population mean.
  $N(\theta, \mu)$, $Exp(1/\theta)$
  $Poisson(\theta)$

- $\theta = \sigma^2$ for a distribution

- θ a range for a distribution
  $U(a, \theta)$

$\hookrightarrow$ Figure out the upper bound.

$\hookrightarrow$ Many more examples

- think about surveying / testing a population

How to do this?

Collect data. - Random sample

$$X_1, X_2, X_3, \dots, X_n \quad - \text{iid.}$$

An estimator

$$\hat{\theta} = \hat{\theta}(X_1, X_2, \dots, X_n).$$

$\hookrightarrow$ function of the data.

Note $\hat{\theta}$ is random

Ex Suppose $\theta = \mathbb{E}X$ (mean of an RV)

A good estimator given $(X_1, X_2, \ldots, X_n)$

$$\hat{\theta} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \swarrow \quad \text{LLN.}$$

↳ there are many more ways to do this.

Questions?. - How good is the estimator?
- Does it get better as we take larger samples?
- Can we guess how far off we are?

Bias, Mean square Error

On average, how far away is $\hat{\theta}$ from θ?

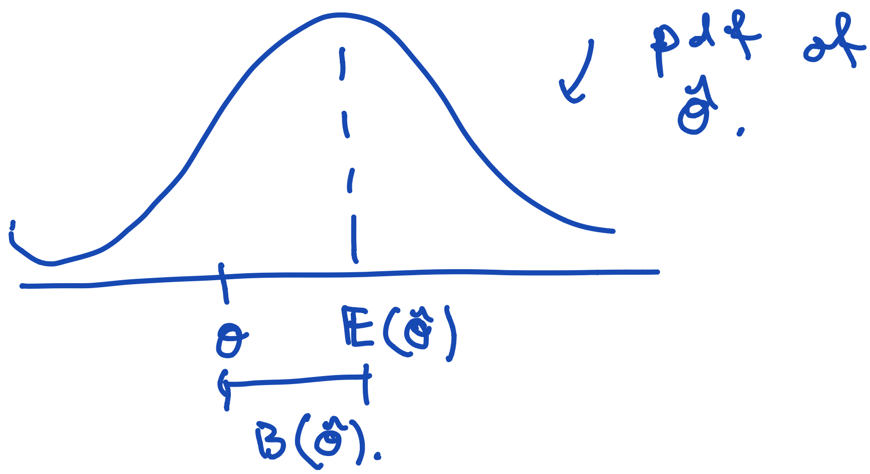
Bias the Bias $B(\hat{\theta})$ of an estimator $\hat{\theta}$ of θ is

$$B(\hat{\theta}) = \mathbb{E}\hat{\theta} - \theta$$

↳ This assumes you know the parameter.

$\hat{\theta}$ is called an unbiased estimator if.

$$B(\hat{\theta}) = 0 \iff \mathbb{E}\hat{\theta} = \theta$$



Ex $X_1, X_2, \ldots, X_n$ random sample

$\mathbb{E}X_i = \mu = \theta$

$\hat{\theta}_1 = X_1$

$$\hat{\theta}_2 = \bar{X} = 1/n \sum_{i=1}^{n} X_i$$

Both unbiased.

- $\mathbb{E}\hat{\theta}_1 = \mathbb{E}X_1 = \theta \Rightarrow B(\hat{\theta}_1) = 0.$
- $\mathbb{E}\hat{\theta}_2 = \mathbb{E}\bar{X} = 1/n \sum_{i=1}^{n} \mathbb{E}X_i = \frac{\cancel{n}}{\cancel{n}}\theta \Rightarrow B(\hat{\theta}_2) = 0.$

Mean square Error

The mean square error of $\hat{\theta}$ is

$$MSE(\hat{\theta}) = \mathbb{E}(\underbrace{\hat{\theta} - \theta}_{\text{error}})^2$$

Ex $\hat{\theta}_1 = X_1$, $\hat{\theta}_2 = \bar{X}$

Assume $\mathbb{E}X_i = \theta$

$Var(X_i) = \sigma^2$

$$MSE(\hat{\theta}_1) = \mathbb{E}(\hat{\theta}_1 - \theta)^2$$

$$= \mathbb{E}(\hat{\theta}_1 - \underbrace{\mathbb{E}\hat{\theta}_1}_{\text{unbiased.}})^2$$

$$= Var(\hat{\theta}_1) = Var(X_1) = \sigma^2.$$

$$MSE(\hat{\theta}_2) = Var(\hat{\theta}_2) = Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right).$$

$$= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{n}{n^2} \sigma^2$$

$$n\sigma^2$$

$$= \frac{\sigma^2}{n}.$$

$MSE(\hat{\theta}_2) \to 0$ as $n \to \infty$.

$$\underline{MSE(\hat{\theta}_2) < MSE(\hat{\theta}_1)} \quad n \geq 1.$$

One is better than the other.

We saw if $B(\hat{\theta}) = 0$,

then $MSE(\hat{\theta}) = Var(\hat{\theta})$.

What if $B(\hat{\theta}) \neq 0$?

Useful formula

If $\hat{\theta}$ is an estimator for θ, then

$$\boxed{MSE(\hat{\theta}) = Var(\hat{\theta}) + B(\hat{\theta})^2.}$$

Proof $\quad X = \hat{\theta} - \theta \quad , \quad \mathbb{E}(X) = B(\hat{\theta}).$

then

$$Var(X) = \mathbb{E}(X^2) - \underset{\mathbb{E}(X)}{\underset{\shortparallel}{(B(\hat{\theta}))}}^2$$

$$\underset{MSE(\hat{\theta})}{\underset{\shortparallel}{\mathbb{E}(\hat{\theta} - \theta)^2}} = \underset{Var(\hat{\theta})}{\underset{\shortparallel}{Var(\hat{\theta} - \theta)}} + (B(\hat{\theta}))^2$$

then

$$\underline{MSE(\hat{\theta}) = Var(\hat{\theta}) + (B(\hat{\theta}))^2.}$$

Examples

① Estimate $\theta = \mu$ of a population

$$\hat{\theta} = \bar{X} = 1/n \sum_{i=1}^{n} X_i \quad \text{— unbiased.}$$

$$\sigma^2_{\hat{\theta}} = Var(\hat{\theta}) = \sigma^2/n$$

② Estimating the fraction p of a population with a certain property (survey).

$$\hat{\theta} = \hat{p} = \frac{X}{n}, \quad X = \text{\# of members of the pop. who have the property out of } n \text{ sampled}$$

↳ unbiased.

$$\sigma^2_{\hat{p}} = Var(\hat{\theta}) = \frac{p(1-p)}{n}$$

↳ if the member of the population are independently sampled.

Estimating the variance

$$\sigma^2 = \mathbb{E}(X-\mu)^2$$

This is just an RV.

$$Y = (X-\mu)^2$$

What is an estimator for σ^2?

How about:

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$$

↳ This is unbiased.

$$\mathbb{E}\,\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\overbrace{\mathbb{E}(X_i-\mu)^2}^{\text{“}\sigma^2\text{”}} = \sigma^2$$

However, in general, we don't know μ.

Lets use $\bar{X}$ instead.

$$\bar{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

What is its bias?

$$\bar{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

"$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2$"

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X})^2$$

therefore

$$\mathbb{E}\,\bar{S}^2 = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}X_i^2 - \mathbb{E}(\bar{X})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(\underbrace{Var(X_i)}_{=\sigma^2} + \mu^2) - [\underbrace{Var(\bar{X})}_{=\sigma^2/n} + \mu^2]$$

$$= \sigma^2 + \cancel{\mu^2} - \frac{\sigma^2}{n} - \cancel{\mu^2}$$

$$= (1 - 1/n)\,\sigma^2.$$

$$B(\bar{S}^2) = \mathbb{E}(\bar{S}^2) - \sigma^2 = -\frac{\sigma^2}{n}.$$

↳ Not unbiased!

Underestimates σ^2.

How to find an unbiased estimator?

$$\mathbb{E}(\bar{s}^2) = (1 - 1/n)\sigma^2 = \left(\frac{n-1}{n}\right)\sigma^2$$

$$\Rightarrow \mathbb{E}\left(\frac{n}{n-1}\bar{s}^2\right) = \sigma^2$$

$$s^2 = \frac{n}{n-1}\bar{s}^2 = \frac{\cancel{n}}{n-1}\frac{1}{\cancel{n}}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$\boxed{s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

↳ sample variance

↳ unbiased

$$\boxed{\mathbb{E}\, s^2 = \sigma^2.}$$