APMA 1650 - Spring 2021
Lecture 12

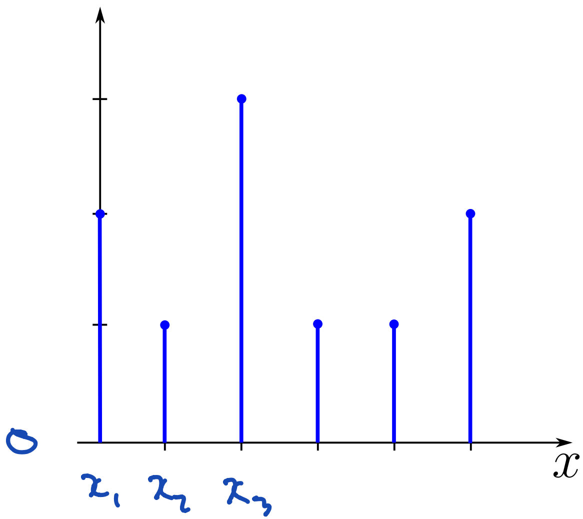
Wednesday, Feb 17, 2021

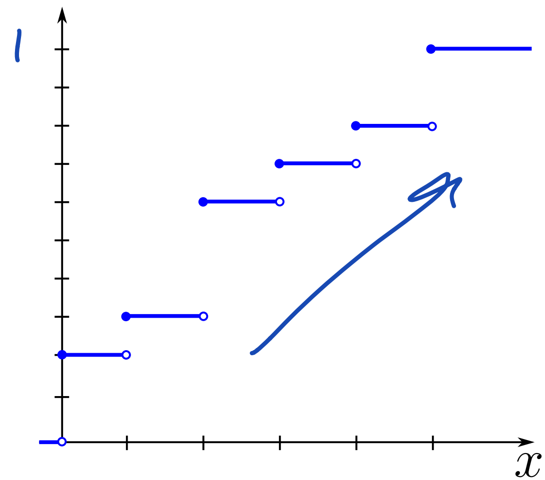
Review of Discrete Random variables

So far we have only dealt with discrete random variables with a countable range $R_X = \{x_1, x_2, \ldots\} \subset \mathbb{R}$.

PMF: $P_X(x) = P(X = x)$ CDF: $F_X(x) = P(X \leq x)$

$x \in \mathbb{R}$





Properties of the CDF

Recall the CDF satisfies the following properties:

1. $F_X(-\infty) = \lim_{x\to-\infty} F_X(x) = 0$
2. $F_X(\infty) = \lim_{x\to\infty} F_X(x) = 1$
3. If $x \leq y$ then $F_X(x) \leq F_X(y)$ (non decreasing)

These properties hold for ANY random variable X

Continuous Probability

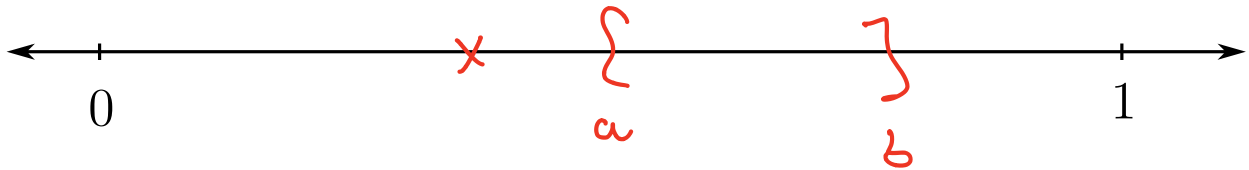
Not all sample spaces/ random variables are naturally discrete. Sometimes their range can be an uncountable set.

Example:

- The amount of rain fall in a storm
- The amount of time it takes for a light bulb to burn out
- The temperature in the room.
- The voltage across a resistor.

A simple example

Pick a random point X in $[0, 1]$ uniformly, $\boxed{X \sim \text{Uniform}(0, 1).}$



Note $R_X = [0, 1]$ is not countable.

For $a < b \in [0, 1]$ we have

$$P(X \in [a, b]) = b - a \quad \leq 1$$

The probability of the point being in $[a, b]$ only depends on the "length" of $[a, b]$.

A simple example

Since a point has no length

$$P(\{X = x\}) = 0, \text{for all } x \in [0, 1].$$

No way to define PMF. All points have zero probability!

$x-\epsilon$

Proof: Since for each $\epsilon > 0$,

x x $x+\epsilon$

$$\{X = x\} \subseteq \{x - \epsilon \leq X \leq x + \epsilon\}$$

then

$$P(X = x) \leq P(x - \epsilon \leq X \leq x + \epsilon) = 2\epsilon$$

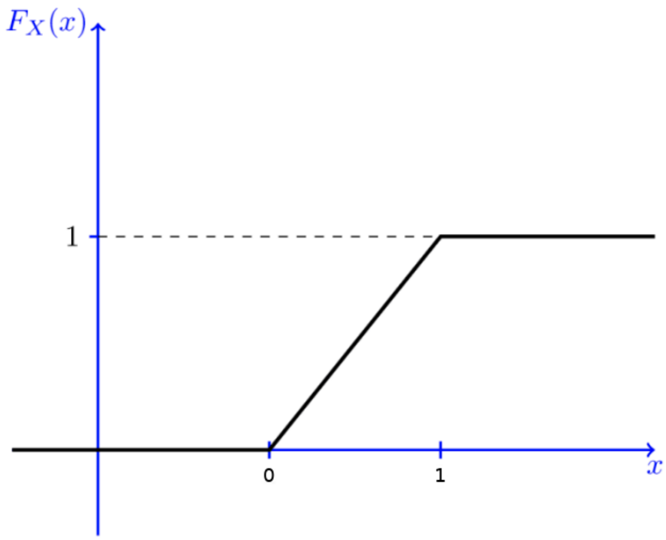
Sending $\epsilon \to 0$ gives $P(X = x) = 0$.

A simple example

However the CDF is well defined for any random variable

$$P(X \leq x) = P(X \in [0, x]) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

CDF has no jumps here! It is completely continuous.

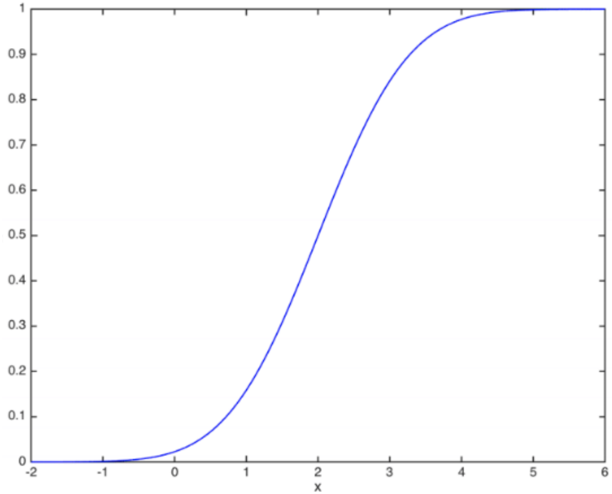


Continuous Random Variables

Let X be any random variable, then it's CDF is still defined by

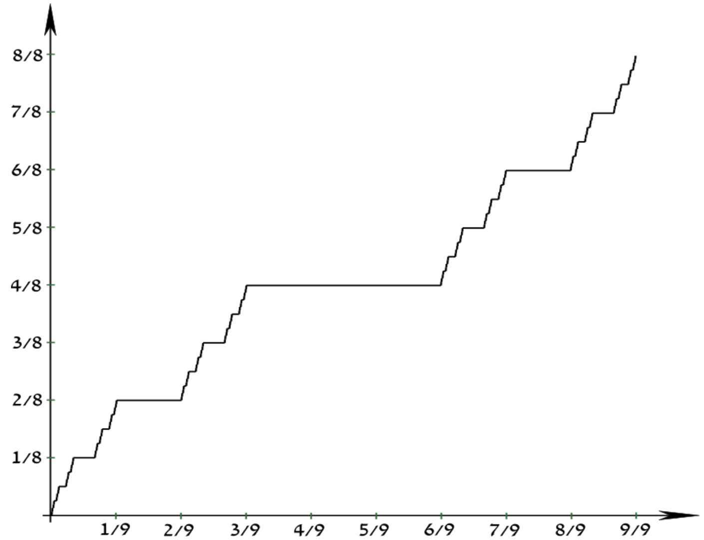
$$F_X(x) := P(X \leq x).$$

Definition: A random variable is said to be a continuous random variable if it's CDF $F_X(x)$ is an absolutely continuous function on $\mathbb{R}$.



What the heck is absolutely continuous?

Not all continuous CDFs are nice



This is the CDF of a random variable that uniformly takes values in the Cantor set (an uncountable fractal subset). The CDF is called the Devil's stair case.

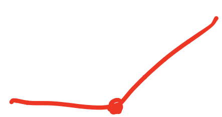
https://en.wikipedia.org/wiki/Cantor_distribution

Absolute Continuity

A function F is absolutely continuous if it is differentiable "almost everywhere" and also satisfies the FTOC

$$F(b) - F(a) = \int_a^b F'(x)\mathrm{d}x.$$

For our purposes it suffices to consider functions which are continuous everywhere and differential except at a countable number of points



not differentiable

Probability density

PMF is not well defined because there is no probability at each point. Instead we could define a probability density

$$f_X(x) = \lim_{\epsilon \to 0} \frac{P(x < X \leq x + \epsilon)}{\epsilon}$$

This is density in the sense that it is a probability per measurement unit on $\mathbb{R}$.
Note

$$P(x < X \leq x + \epsilon) = F_X(x + \epsilon) - F_X(x)$$

therefore

$$f_X(x) = \lim_{\epsilon \to 0} \frac{F_X(x + \epsilon) - F_X(x)}{\epsilon} = F'_X(x)$$

Probability density function (PDF)

Definition Let X be a continuous random variable with an absolutely continuous CDF $F_X(x)$, then

$$f_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x} = F'_X(x)$$

is called the probability density function (PDF) of X and satisfies

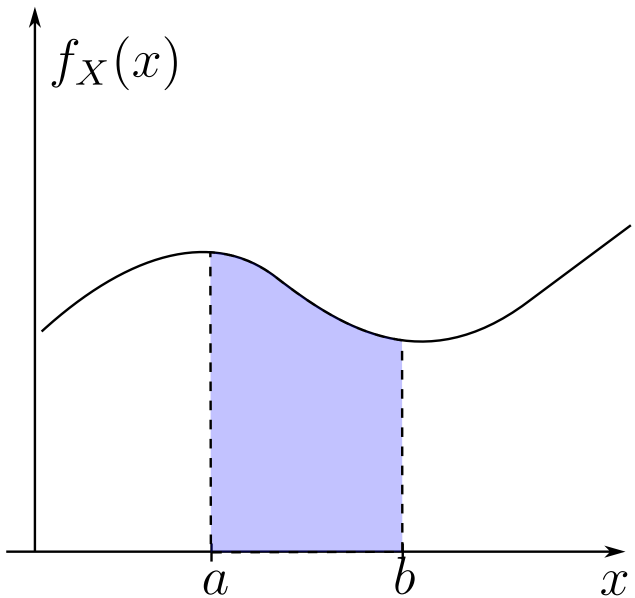
$$F_X(x) = \int_{-\infty}^{x} f_X(x)\mathrm{d}x.$$

FTOC

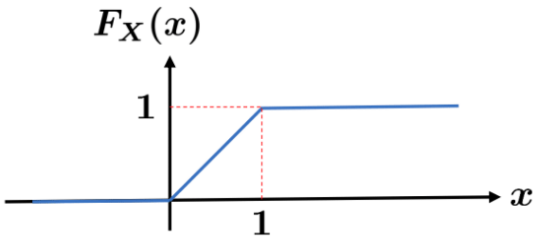
Area rule

$P(X \leq b) - P(X \leq a)$

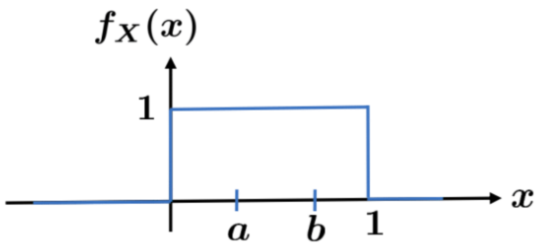
$$
\begin{aligned}
P(a < X \leq b) &= F_X(b) - F_X(a) \\
&= \int_{-\infty}^{b} f_X(x)\mathrm{d}x - \int_{-\infty}^{\cancel{b}\ a} f_X(x)\mathrm{d}x \\
&= \int_{a}^{b} f_X(x)\mathrm{d}x
\end{aligned}
$$



Example: $X \sim \text{Uniform}(0, 1)$



$$\frac{d(.)}{dx} \Rightarrow$$



$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \qquad f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

$$P(a \le X \le b) = \int_a^b f_X(x)\mathrm{d}x = \int_a^b 1\mathrm{d}x = b - a.$$

$<$ $\le$

$\le$ $<$ $\rightarrow$ All sum

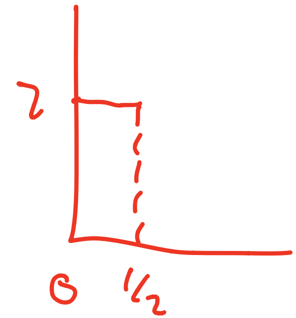
$<$ $<$

Properties of the PDF

1. $f_X(x) \geq 0$ for all $x \in \mathbb{R}$ (since $F_X(x)$ is non-decreasing)

2. $$F_x(\infty) = \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1 \quad \text{(normality)}$$

$$F_x(\infty) = 1$$



PDF can take values bigger than 1 bigger than 1!

We will often define the range of X by the support of f_X

$$R_X = \{x : x \in \mathbb{R}, f(x) > 0\}$$

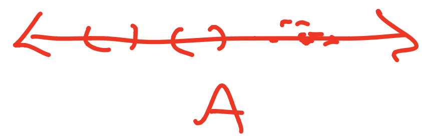
This can miss probability zero values

Properties of the PDF

3. Area rule

$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\mathrm{d}x$$

4. More generally for "any" set $A \subset \mathbb{R}$



$$P(X \in A) = \int_A f_X(x)\mathrm{d}x$$

Here "any" set means any countable union and intersection of intervals.

Example: Let X be a continuous RV with the PDF

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find c, $F_X(x)$, and $P(1 < X < 3)$.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} ce^{-x}\,dx$$

$$\Rightarrow c = 1$$

$$= c\int_0^{\infty} e^{-x}\,dx = c$$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_0^x e^{-y}\,dy = \begin{cases} 1 - e^{-x}, & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$P(1 < X < 3) = \int_1^3 e^{-x}\,dx = e^{-1} - e^{-3}$$

Comparison to Discrete RVs

$$P(X \in A): \quad \underbrace{\sum_{x \in A} P_X(x)}_{\text{discrete}} \quad \Rightarrow \quad \underbrace{\int_A f_X(x)\mathrm{d}x}_{\text{continuous}}$$

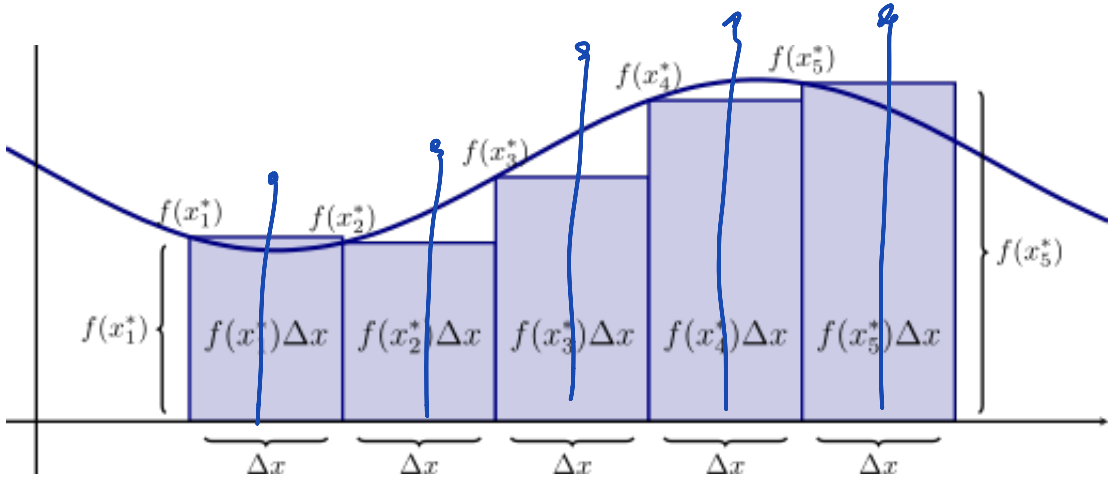
- For discrete we sum the probabilities over all $x \in A$
- For continuous we integrate PDF over the set A

Riemann Sums

Can approximate a continuous RV X by discrete on $X^{(n)}$ using Riemann sums

$$\int_a^b f_X(x)\mathrm{d}x \approx \sum_{i=1}^{n} \underbrace{f_X(x_i^*)\Delta x_i}_{\text{Behaves like a probability}}$$

Choose $X^{(n)}$ to have $P_{X^{(n)}}(x_i^*) = f_X(x_i^*)\Delta x_i$, $X^{(n)}$ is discrete.



Expected Value

Recall for a discrete random variable

$$EX = \sum_{x \in R_X} x P_X(x)$$

By the discrete to continuous analogy given above

$$\sum_{x} \Rightarrow \int_{-\infty}^{\infty}, \qquad P_X(x) \Rightarrow f_X(x)$$

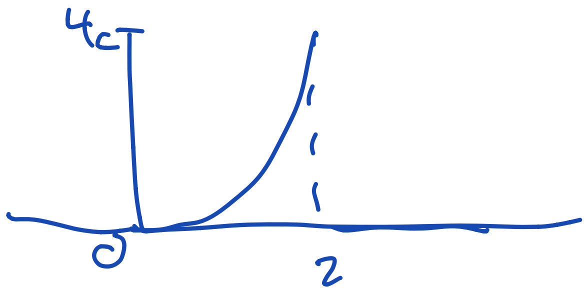
Definition Let X be a continuous random variable, then it's expected value is

$$EX = \mu_X = \int_{-\infty}^{\infty} x \overbrace{f_X(x)\mathrm{d}x}^{\text{"}P_X(x)\text{"}}$$

Example: Consider a random variable X with PDF

$$f_X(x) = \begin{cases} c\overset{x^2}{\cancel{y}}{}^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find c and find $E[X]$.



$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = c\int_0^2 x^2\,dx = c\,\tfrac{1}{3}\left(x^3\Big|_0^2\right)$$

$$= c\,\tfrac{8}{3}$$

$$c = \tfrac{3}{8}$$

$$EX = \int_{-\infty}^{\infty} x f_X(x)\,dx = \frac{3}{8}\int_0^2 x(x^2)\,dx$$

$$= \frac{3}{8}\int_0^2 x^3\,dx$$

$$= \frac{3}{8}\,\frac{1}{4}\,x^4 \Big|_0^2$$

$$= \frac{3}{8}\,\frac{16}{4} = \frac{3}{2}$$

LOTUS

We also have the natural extension of the law of the unconscious statistician for continuous random variables.

LOTUS Given a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \mathrm{d}x.$$

Linearity of Expectation

Linearity of expectation is a general property of ANY random variable and still holds: For $a, b \in \mathbb{R}$

$$E[aX + b] = aE[X] + b$$

Proof:

$$\underbrace{\int_{-\infty}^{\infty} (ax + b) f_X(x) \mathrm{d}x}_{E[aX+b]} = a \underbrace{\int_{-\infty}^{\infty} x f_X(x) \mathrm{d}x}_{E[X]} + b \underbrace{\int_{-\infty}^{\infty} f_X(x) \mathrm{d}x}_{1}$$

More generally for $X_1, X_2, \ldots X_n$ continuous RVs we still have

$$E[X_1 + X_2 + \ldots X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

Variance

As in the case with discrete random variables, we are interested in average deviations from the mean. Choose $g(x) = (x - \mu_X)^2$.

Definition: The variance of a continuous random variable X is defined by

$$\mathrm{Var}(X) = \sigma_X^2 = E(X - \mu_X)^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) \mathrm{d}x.$$

The following formula still holds! (follows only from linearity)

$$\mathrm{Var}(X) = EX^2 - (EX)^2$$

Example Consider a random variable X with PDF

$$f_X(x) = \begin{cases} ce^{-x/10} & 0 \leq x \\ 0 & \text{otherwise} \end{cases}$$

Find c, $E[X]$,Var(X) and $E[e^{-X/10}]$.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = c\int_0^{\infty} e^{-x/10}\,dx = -10c\left(e^{-x/10}\right|_0^{\infty}$$

$$= 10c \Rightarrow c = 1/10$$

$$EX = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \frac{1}{10}\int_0^{\infty} x e^{-x/10}\,dx$$

IBP

$$= \underbrace{-xe^{-x/10}\Big|_0^{\infty}}_{=0} + \int_0^{\infty} e^{-x/10}\,dx = 10$$

$$\mathrm{Var}(X) = EX^2 - (EX)^2$$

$$EX^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \tfrac{1}{10}\int_0^{\infty} x^2 e^{-x/10}\,dx$$

$$= \underbrace{-x^2 e^{-x/10}\Big|_0^{\infty}}_{=0} + 2\int_0^{\infty} x\, e^{-x/10}\,dx$$

$$= \underbrace{-20x\, e^{-x/10}\Big|_0^{\infty}}_{=0} + \int_0^{\infty} 20\, e^{-x/10}\,dx = -20\times 10\, e^{-x/10}\Big|_0^{\infty}$$

$$= 200$$

$$\Rightarrow \quad \mathrm{Var}(X) = 200 - 10^2 = 200 - 100 = 100.$$

$$\mathbb{E}\, e^{-X/10} = \frac{1}{10}\int_0^{\infty} e^{-x/10} e^{-x/10}\,dx = \tfrac{1}{10}\int_0^{\infty} e^{-x/5}\,dx = \frac{5}{10} = \frac{1}{2}.$$

Summary

- PDF: $f_X(x) = F_X'(x)$, $f_X(x) \geq 0$, $\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1$
- Expected Value $EX = \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x$
- LOTUS: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\mathrm{d}x$
- Variance: $\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\mathrm{d}x$
- Area Rule:
  $P(a < X < b) = \int_a^b f_X(x)\mathrm{d}x = F_X(b) - F_X(a)$ or
  $P(X \in A) = \int_A f_X(x)\mathrm{d}x$.