# APMA 1650 - Spring 2021
# Lecture 14

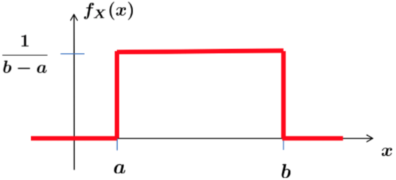
Wed, Feb 24, 2021

# Uniform Distribution

**Uniform Distribution:** $X \sim \text{Uniform}(a, b), \quad a < b$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$



# Uniform Distribution

**Expectaion and Variance**: If $X \sim \text{Uniform}(a, b)$, then

$$EX = \frac{b+a}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}$$

$$EU = \frac{1}{2}, \qquad \mathsf{Var}(U) = \frac{1}{12}$$

Neat trick: Rescale

$$U = \frac{X-a}{b-a} \sim \mathsf{Uniform}(0,1).$$

Then

$$X = (b-a)U + a$$

$$\Downarrow$$

$$EX = (b-a)EU + a, \quad \mathsf{Var}(X) = (b-a)^2 \mathsf{Var}(U)$$

Therefore

$$EX = \frac{(b-a)}{2} + a = \frac{b+a}{2}, \quad \mathsf{Var}(X) = \frac{(b-a)^2}{12}$$

# Exponential Distribution

**Exponential Distribution:** $X \sim \text{Exponential}(\lambda)$

- Very common distribution
- Measure random waiting "time" between rare events
  - Random alarm clock
  - Time between arrivals at a service center
  - Time between emails arriving
  - Time between spike inputs to a neuron
- $\lambda$ is a rate describing how frequently these events happen
- Related to Poisson distribution (same rate $\lambda$)
- A continuous version of the geometric distribution

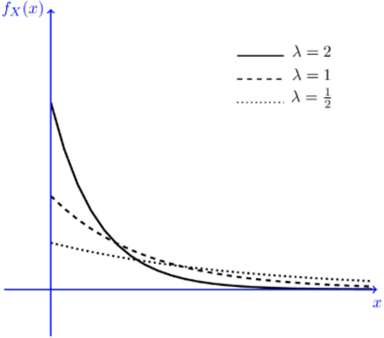
# Exponential Distribution

**Exponential Distribution:** $X \sim \text{Exponential}(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= \lambda e^{-\lambda x} u(x)$$

where $u(x)$ is the unit step function

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



# Exponential Distribution

**CDF**

$$F_X(x) = (1 - e^{-\lambda x})u(x)$$

# Expected Value

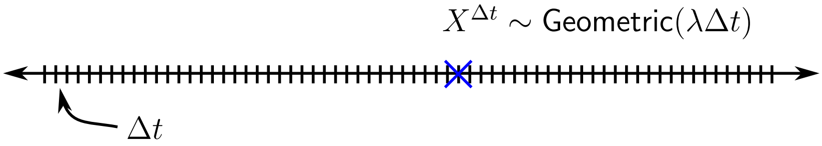
$$EX = \frac{1}{\lambda}$$

# Variance

$$\mathsf{Var}(X) = \frac{1}{\lambda^2}$$

# Relation to Geometric distribution

We can think of an exponential random variable as a continuous geometric distribution with rare success $p = \lambda \Delta t$.




Let $x = k\Delta t$ and

$$\begin{aligned} P(X^{\Delta t} \leq k) &= 1 - (1 - \lambda \Delta t)^k \\ &= 1 - (1 - \lambda \Delta t)^{x/\Delta t} \\ &\to 1 - e^{-\lambda x} \quad \text{as} \quad \Delta t \to 0 \end{aligned}$$

$$\frac{X^{\Delta t}}{\Delta t} \approx \text{Exponential}(\lambda)$$

# Memoryless Property

- $X \sim$ Exponential$(\lambda)$ is a memoryless random variable, just like the geometric
- An exponential alarm clock doesn't change it's likelihood of going off given that you know it hasn't gone off yet
- No matter how long you have already waited $X > a$, the probability that you have to wait $x$ amount longer is always the same $P(X > x)$.

$$P(X > x + a | X > a) = P(X > x), \quad a, x \geq 0$$

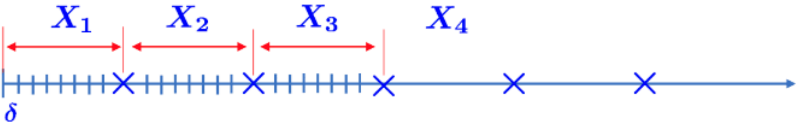
# Memoryless proof

**Proof**:

$$
\begin{aligned}
P(X > x + a | X > a) &= \frac{P(X > x + a, X > a)}{P(X > a)} \\
&= \frac{P(X > x + a)}{P(X > a)} \\
&= \frac{1 - F_X(x + a)}{1 - F_X(a)} \\
&= \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} \\
&= e^{-\lambda x} \\
&= P(x > x)
\end{aligned}
$$

Exponential is the ONLY memoryless continuous RV

# Relation to Poisson Distribution




- Time differences between rare events are independent Exponential($\lambda$), $X_1, X_2, \ldots$
- $Y \sim$ Poisson($\lambda t$). Number of events happening in $[0, t]$

$$X = \text{first rare event time}$$

$$\begin{aligned} P(X > t) &= P(\text{no events in } [0,t]) = P_Y(0) \\ &= e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t} \\ &= 1 - F_X(t) \end{aligned}$$