APMA 1650 - Spring 2021
Lecture 14

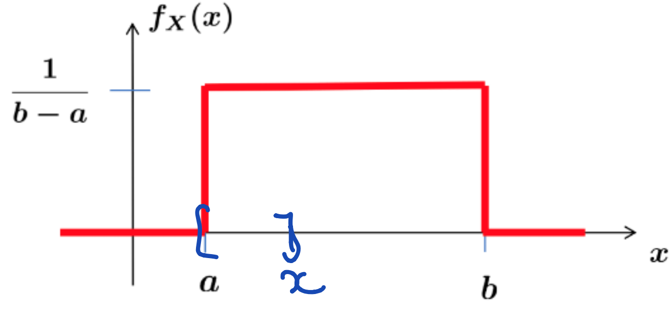
Wed, Feb 24, 2021

Uniform Distribution

$$\int_a^b \frac{1}{b-a}\,dx = \frac{b-a}{b-a} = 1$$

Uniform Distribution: $X \sim \text{Uniform}(a, b), \quad a < b$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$



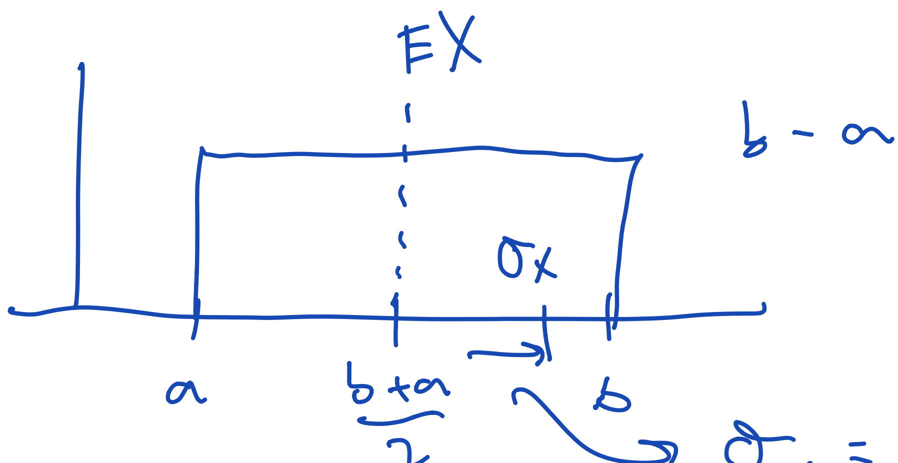
$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$P(X \in [a, x]) = \frac{x-a}{b-a}$$

Uniform Distribution

Expectaion and Variance: If $X \sim \text{Uniform}(a, b)$, then

$$EX = \frac{b+a}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}$$



$$\sigma_X = \frac{b-a}{2\sqrt{3}}$$

$$EU = \frac{1}{2}, \qquad \mathsf{Var}(U) = \frac{1}{12}$$

$U \sim \text{Uniform}(0,1)$. - standard uniform

$$\mathbb{E}U = \int_0^1 u \times 1 \, du = \tfrac{1}{2} u^2 \Big|_0^1 = 1/2$$

$$\mathbb{E}U^2 = \int_0^1 u^2 \times 1 \, du = \tfrac{1}{3} u^3 \Big|_0^1 = 1/3$$

$$\mathrm{Var}(U) = \mathbb{E}U^2 - (\mathbb{E}U)^2 = 1/3 - (1/2)^2 = 1/3 - 1/4 = 1/12$$

$$f_U(u) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Neat trick: Rescale

$X \sim \text{Uniform}(a,b)$

$= F_X(x)$

$$U = \frac{X-a}{b-a} \sim \text{Uniform}(0,1).$$

Then

$$X = (b-a)U + a \quad \sim \text{Uniform}(a,b)$$

$$\Downarrow$$

$$EX = (b-a)EU + a, \quad \text{Var}(X) = (b-a)^2\text{Var}(U)$$

Therefore

$$EX = \frac{(b-a)}{2} + a = \frac{b+a}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}$$

Exponential Distribution

Exponential Distribution: $X \sim \text{Exponential}(\lambda)$

$T \sim$

- Very common distribution
- Measure random waiting "time" between rare events
  - Random alarm clock
  - Time between arrivals at a service center
  - Time between emails arriving
  - Time between spike inputs to a neuron
- λ is a rate describing how frequently these events happen
- Related to Poisson distribution (same rate λ)
- A continuous version of the geometric distribution

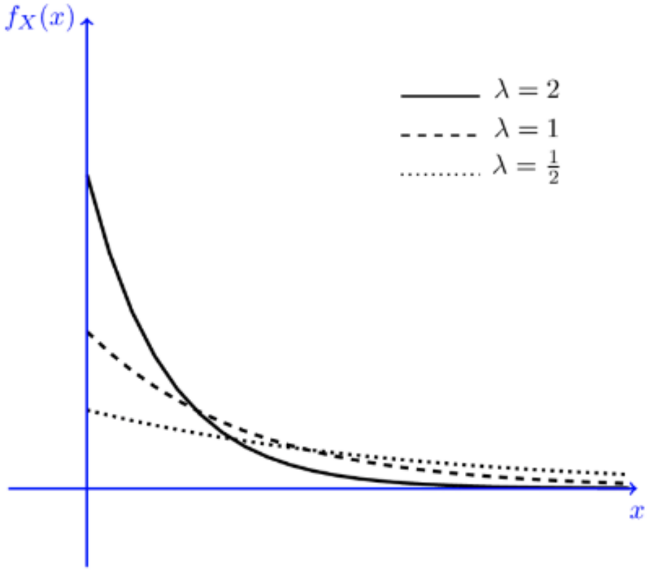
Exponential Distribution

Exponential Distribution: $X \sim \text{Exponential}(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= \lambda e^{-\lambda x} u(x)$$

where $u(x)$ is the unit step function

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Exponential Distribution

CDF

$$F_X(x) = (1 - e^{-\lambda x})u(x)$$

$x \geq 0$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_0^x \lambda e^{-\lambda u}\,du.$$

$$= -\frac{\lambda}{\lambda} e^{-\lambda u}\Big|_0^x = -\left(e^{-\lambda x} - \underset{1}{e^0}\right)$$

$$= \left(1 - e^{-\lambda x}\right)u(x)$$

$$\lim_{x \to \infty} F_X(x) = \lim_{x \to \infty} \left(1 - e^{-\lambda x}\right) = 1$$

Expected Value

$X \sim \text{Exponential}(\lambda)$.

$$EX = \frac{1}{\lambda}$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

integration by parts

$$= \lambda \int_0^\infty \underbrace{x}_{u} \underbrace{e^{-\lambda x}dx}_{dv} = \underbrace{-\underbrace{x}_{u}\underbrace{e^{-\lambda x}}_{v}\Big|_0^\infty}_{=0} - \lambda\int_0^\infty \underbrace{1}_{du}\underbrace{\left(-\frac{1}{\lambda}e^{-\lambda x}\right)}_{v}dx$$

$$= \int_0^\infty e^{-\lambda x}\,dx = -\frac{1}{\lambda}e^{-\lambda x}\Big|_0^\infty = 1/\lambda.$$

Variance

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

$$\mathbb{E}X^2 = \int_0^\infty \underbrace{x^2}_{u} \underbrace{\lambda e^{-\lambda x}}_{dv} dx = \underbrace{-x^2 e^{-\lambda x}\Big|_0^\infty}_{0} - \int_0^\infty 2x\left(-e^{-\lambda x}\right)dx$$

$$= \int_0^\infty 2x e^{-\lambda x} dx = \frac{2}{\lambda}\underbrace{\left(\int_0^\infty x \lambda e^{-\lambda x} dx\right)}_{\mathbb{E}X = 1/\lambda}$$

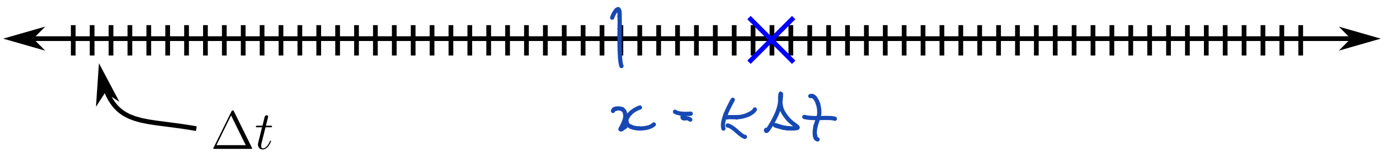
$$= 2/\lambda^2$$

$$\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 2/\lambda^2 - \left(1/\lambda^2\right) = 1/\lambda^2$$

Relation to Geometric distribution

We can think of an exponential random variable as a continuous geometric distribution with rare success $p = \lambda \Delta t$.

$X^{\Delta t} \sim \text{Geometric}(\lambda \Delta t)$



Let $x = k\Delta t$ and

$$P(X^{\Delta t} \leq k) = 1 - (1 - \overbrace{\lambda \Delta t}^{p})^k$$

$$= 1 - (1 - \lambda \Delta t)^{x/\Delta t}$$

$= \frac{x}{\Delta t}$

$$\to 1 - e^{-\lambda x} \quad \text{as} \quad \Delta t \to 0$$

logarithm

↓ L'Hôpital.

$$\Delta t \, \frac{X^{\Delta t}}{\cancel{\Delta t}} \approx \text{Exponential}(\lambda)$$

Memoryless Property

- $X \sim$ Exponential(λ) is a memoryless random variable, just like the geometric
- An exponential alarm clock doesn't change it's likelihood of going off given that you know it hasn't gone off yet
- No matter how long you have already waited $X > a$, the probability that you have to wait x amount longer is always the same $P(X > x)$.

$$P(X > x + a | X > a) = P(X > x), \quad a, x \geq 0$$

Memoryless proof

Proof:

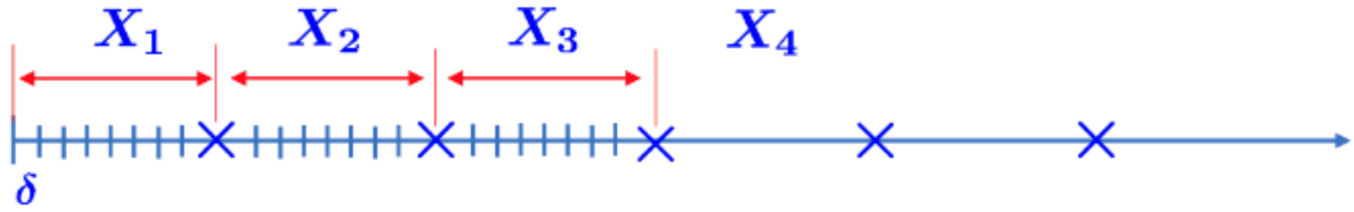
$$
\begin{aligned}
P(X > x + a | X > a) &= \frac{P(X > x + a, X \cancel{>} a)}{P(X > a)} \\
&= \frac{P(X > x + a)}{P(X > a)} \\
&= \frac{1 - F_X(x + a)}{1 - F_X(a)} \\
&= \frac{e^{-\lambda(x+\cancel{a})}}{e^{-\lambda \cancel{a}}} \\
&= e^{-\lambda x} \\
&= P(\cancel{x} > x)
\end{aligned}
$$

$P(X > x)$

Survival function,

Exponential is the ONLY memoryless continuous RV

Relation to Poisson Distribution



- Time differences between rare events are independent Exponential(λ), $X_1, X_2, \ldots$
- $Y \sim$ Poisson(λt). Number of events happening in $[0, t]$

$$X = \text{first rare event time}$$

$$\begin{aligned} P(X > t) &= P(\text{no events in } [0,t]) = P_Y(0) \\ &= e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t} \\ &= 1 - F_X(t) \end{aligned}$$

exponential.