APMA 1650 - Spring 2021
Lecture 15

Fri, Feb 26, 2021

Announcements

- Fill out the Midterm feedback form (link in Campuswire)

Normal Distribution

Normal (Gaussian) Distribution $X \sim N(\mu, \sigma)$

- Two parameters μ is the mean, σ standard deviation.
- Probably THE single most important distribution in existence
- Arises naturally from the Central Limit Theorem (CLT)
- Large sums of independent identically distributed random variables can be approximated by Normal

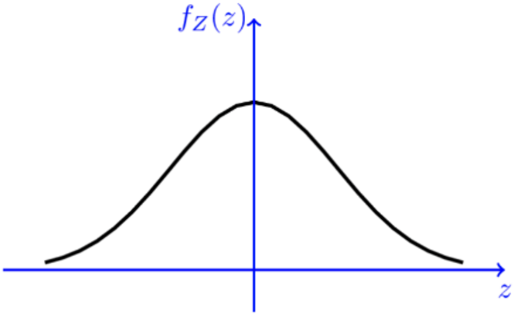
$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \approx N(0,1) \quad EX_i = 0, \mathsf{Var}(X_i) = 1$$

More on this later!

Standard Normal

Standard Normal $Z \sim N(0,1)$.

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z \in \mathbb{R}$$



Why is the Normal normalized?

How do you show that

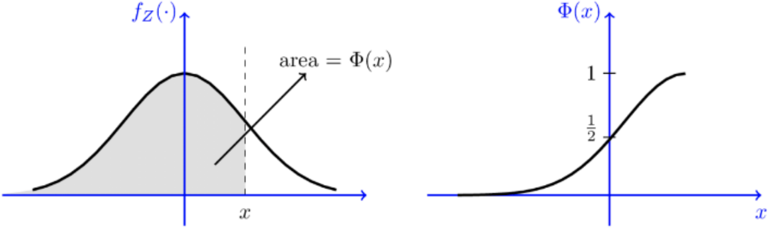
$$\int_{-\infty}^{\infty} e^{-x^2/2} \mathrm{d}x = \sqrt{2\pi}?$$

- It's not easy (simplest proof I know requires multivariable)
- $e^{-x^2/2}$ doesn't have a nice anti-derivative (can't be written in terms of elementary functions)

Normal CDF

The CDF of the standard normal:

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \mathrm{d}z$$



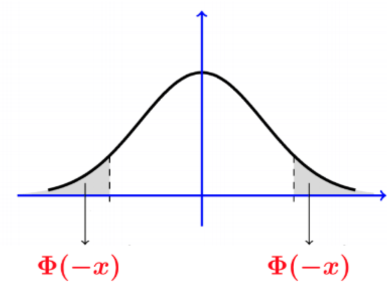
No simple closed form expression. Use CDF calculator!

Ex:`normalcdf(z)` in MATLAB

Symmetry properties of CDF

The CDF of the standard normal has the following properties

1. $\lim_{x\to\infty}\Phi(x) = 1$,
   $\lim_{x\to-\infty}\Phi(x) = 0$
2. $\Phi(0) = 1/2$
3. $\Phi(-x) = 1 - \Phi(x) =$

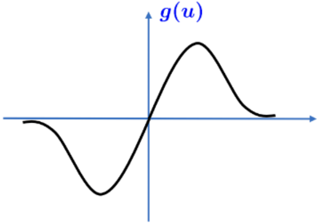


Expected value

$$E[Z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{ue^{-u^2/2}}_{g(u)} \mathrm{d}u$$

$$= 0$$

Since $g(u)$ is an odd function



Variance

$$\mathsf{Var}(Z) = E[Z^2] - \underbrace{(E[Z])^2}_{=0} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} \mathrm{d}z$$

Integration by parts

$$u = z, dv = ze^{-\frac{z^2}{2}} \mathrm{d}z$$

$$du = \mathrm{d}z, \quad v = -e^{-\frac{z^2}{2}}$$

Therefore

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} \mathrm{d}z = \frac{1}{\sqrt{2\pi}} \underbrace{-ze^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty}}_{=0} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \mathrm{d}z = 1$$

Standard normal summary

Mean and variance of standard Normal

$$Z \sim N(0,1) \quad \Rightarrow \quad \begin{cases} \text{Mean: } E[Z] = 0 \\ \text{Variance: } \mathrm{Var}(Z) = 1 \end{cases}$$

Symmetry

$$\Phi(-x) = 1 - \Phi(x), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \mathrm{d}z$$

General Normal random variable

Let $Z \sim N(0,1)$, then for $\sigma > 0, \mu \in \mathbb{R}$

$$X = \sigma Z + \mu \sim N(\mu, \sigma)$$

Note

$$E[X] = E[\sigma Z + \mu] = \sigma \underbrace{EZ}_{=0} + \mu = \mu$$

and

$$\mathsf{Var}(X) = \mathsf{Var}(\sigma Z + \mu) = \sigma^2 \underbrace{\mathsf{Var}(Z)}_{=1} = \sigma^2$$

CDF and PDF

Let $X \sim N(\mu, \sigma)$. Note that

$$X = \sigma Z + \mu \leq x \quad \Leftrightarrow \quad Z \leq \frac{x - \mu}{\sigma}$$

So

$$F_X(x) = P(\sigma Z + \mu \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

and

$$\begin{aligned} f_X(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\Phi\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{\sigma} f_Z\left(\frac{x - \mu}{\sigma}\right) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) \end{aligned}$$

General Normal summary

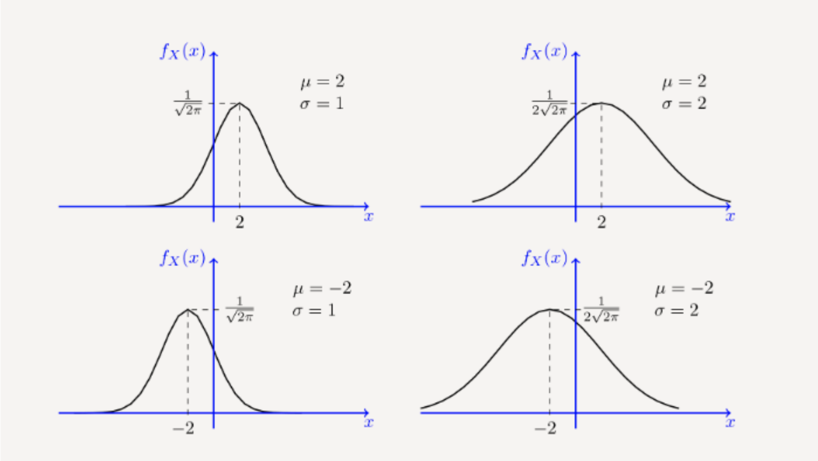
$X \sim N(\mu, \sigma)$,

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$EX = \mu, \quad \mathsf{Var}(X) = \sigma^2$$

$$P(a < X < b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Shape



$f_X(x)$
$\mu = 2$
$\sigma = 1$
$\frac{1}{\sqrt{2\pi}}$
2
x
$f_X(x)$
$\mu = 2$
$\sigma = 2$
$\frac{1}{2\sqrt{2\pi}}$
2
x
$f_X(x)$
$\mu = -2$
$\sigma = 1$
$\frac{1}{\sqrt{2\pi}}$
-2
x
$f_X(x)$
$\mu = -2$
$\sigma = 2$
$\frac{1}{2\sqrt{2\pi}}$
-2
x

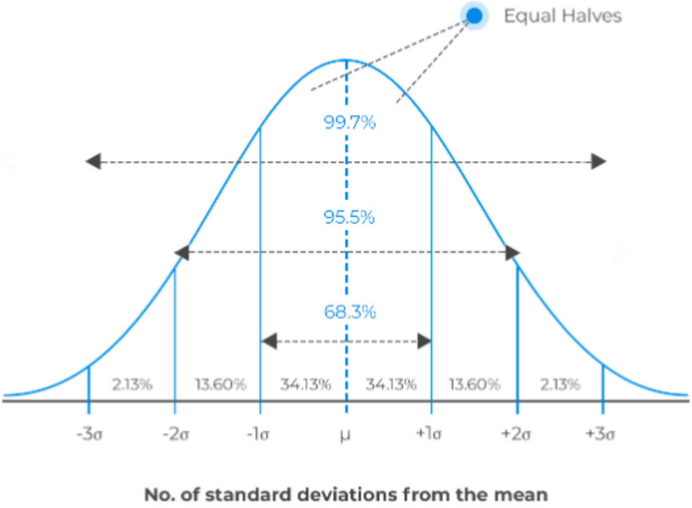
Example

Let $X \sim N(2, 0.5)$. What is $P(1 < X \leq 3)$?

Deviations from the mean

Let $X \sim N(\mu, \sigma^2)$. What is the probability that X is within one σ of the mean? How about 2σ

Deviations from the mean



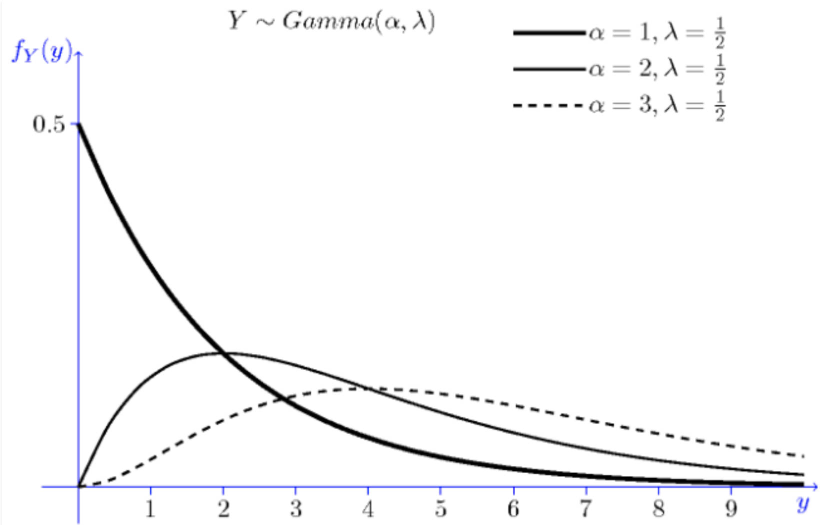
Gamma Distribution

Gamma Distribution: $X \sim \text{Gamma}(\alpha, \lambda)$, $\alpha, \beta > 0$

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- $\Gamma(\alpha)$ is a normalization factor called the Gamma function
- Covers a range of very important distributions for different choice of $\alpha, \beta > 0$
- Very important in inferential statistics, particularly hypothesis testing
- α is often called the shape parameter, and λ the rate

Shape

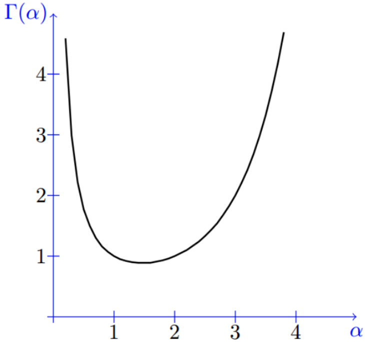


$Y \sim Gamma(\alpha, \lambda)$
$\alpha = 1, \lambda = \frac{1}{2}$
$\alpha = 2, \lambda = \frac{1}{2}$
$\alpha = 3, \lambda = \frac{1}{2}$
$f_Y(y)$
0.5
1
2
3
4
5
6
7
8
9
y

Gamma function

Normalization factor

$$\Gamma(\alpha) = \int_0^\infty \lambda^\alpha x^{\alpha-1} e^{-\lambda x} \mathrm{d}x$$
$$= \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x$$



- Generalization of the factorial to non-integers.
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
- $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$

Relation to Exponential

If $\alpha = 1$,

$$f_X(x) = \lambda e^{-\lambda}, x > 0$$

Therefore

$$\text{Gamma}(1, \lambda) = \text{Exponential}(\lambda)$$

Relation of Pascal

If $X_1, X_2, \dots X_n$ are independent Exponential(λ), then

$$X_1 + X_2 + \dots X_n \sim \Gamma(n, \lambda).$$

Continuous version of Pascal (negative Binomial)

$$EX = \frac{\alpha}{\lambda}, \quad \mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$$

Magically extends to any $\alpha > 0$