# APMA 1650 - Spring 2021
# Lecture 16

Mon, Mar 1st, 2021

# Mixed Random Variables

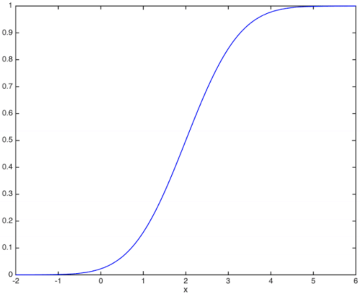
Today we will

- Introduce the generalized PDF via the delta function
- Introduce mixed random variables

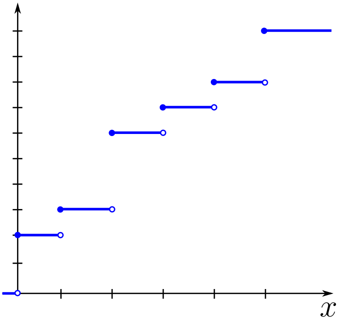
# Mixed Random Variables

Recall the CDF for continuous vs discrete RV $X$

$$F_X(x) := P(X \leq x)$$




Continuous
Differentiable

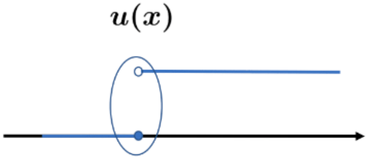



Discrete
Not Differentiable

# Lets take the derivative anyway

Consider the unit step function

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$




What is the derivative?

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x) = \delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

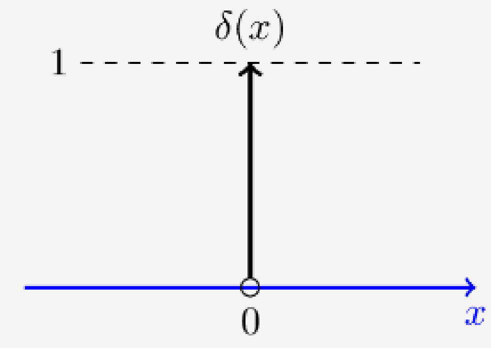
Delta "function" Not really a function.

# Delta function

Note, by the "FTC" that

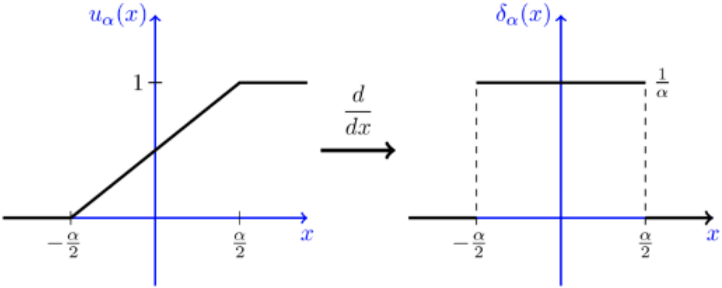
$$\text{“}\int_{-\epsilon}^{\epsilon} \delta(x)\mathrm{d}x\text{”} = \int_{-\epsilon}^{\epsilon} \frac{\mathrm{d}}{\mathrm{d}x} u(x)\mathrm{d}x = u(\epsilon) - u(-\epsilon) = 1$$

So it is like a PDF will all the probability at zero



# Approximating the delta function

Best understood by approximation $u(x)$ by $u_\alpha(x)$




$$\delta_\alpha(x) = \frac{\mathrm{d}u_\alpha(x)}{\mathrm{d}x}, \quad \delta_\alpha(x) \to_{\alpha\to 0} \delta(x)$$

$$\int_{-\alpha/2}^{\alpha/2} \delta_\alpha(x)\mathrm{d}x = 1$$

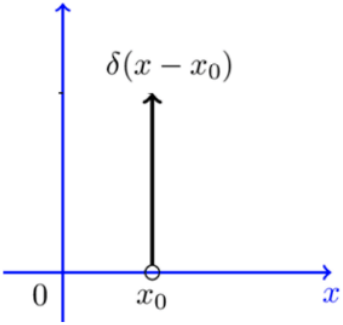
# Moving the delta function around

We can shift the location of the delta function by shifting $u$

$$u(x-x_0) = \begin{cases} 1 & x \geq x_0 \\ 0 & \text{otherwise} \end{cases}$$

then $\delta(x-x_0) = \frac{\mathrm{d}}{\mathrm{d}x} u(x-x_0)$

$$\delta(x-x_0) = \begin{cases} \infty & x = x_0 \\ 0 & \text{otherwise} \end{cases}$$



# The most important property

The delta “function” has an important property that makes it very useful

**The fundamental property of delta functions**

Let $g$ be a continuous function, then

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)\mathrm{d}x = g(x_0).$$

Can replace the integral above with

$$\int_{x_0-\epsilon}^{x_0+\epsilon} g(x)\delta(x - x_0)\mathrm{d}x$$

# Example

Let $g(x) = 2(x^2 + x)$, what is $\int_{-\infty}^{\infty} g(x)\delta(x-1)\mathrm{d}x$?

# Summary

The delta "function" has the following properties

1.
$$\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x}u(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. For any continuous $g(x)$, $x_0 \in \mathbb{R}$ and $\epsilon > 0$.
$$\int_{x_0-\epsilon}^{x_0+\epsilon} g(x)\delta(x - x_0)\mathrm{d}x = g(x_0).$$

Note that choosing $g(x) = 1$ implies that

$$\int_{-\infty}^{\infty} \delta(x - x_k)\mathrm{d}x = 1.$$

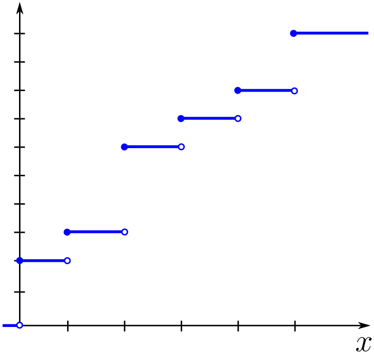
# Back to discrete random variables

Consider a discrete RV $X$ with range $R_X = \{x_1, x_2, x_3, \ldots\}$

We can always write it's CDF as a bunch of jumps

$$F_X(x) = \sum_{x_k \in R_X} P(x_k) u(x - x_k)$$

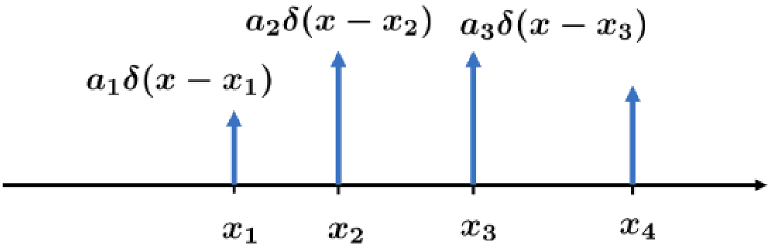
Why is this?



# Generalized PDF

Lets take the derivative of that CDF

$$
\begin{aligned}
f_X(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{x_k \in R_X} P_X(x_k) u(x - x_k) \\
&= \sum_{x_k \in R_X} P_X(x_k) \delta(x - x_k)
\end{aligned}
$$



# Summary

Let $X$ be a discrete RV, then its generalized PDF is given by

$$f_X(x) = \sum_{x_k \in R_X} P_X(x_k)\delta(x - x_k).$$

Can you show that this is a valid PDF?

$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1?$$

# Valid PDF?

# Expectation

What happens with the expectation

$$EX = \int_{-\infty}^{\infty} x f_X(x) \mathrm{d}x?$$

# Example

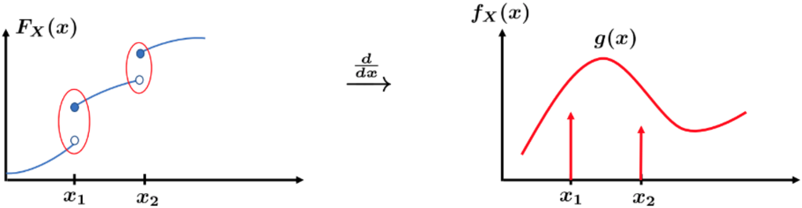
From the HW

$$F_X(x) = \begin{cases} 0 & x < -2 \\ 1/3 & -2 \le x < -1 \\ 1/2 & -1 \le x < 1 \\ 5/6 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

What is $f_X(x)$?

# Mixed Random variables

In general a Random variable can be a mix of discrete and continuous parts




$$\frac{\mathrm{d}}{\mathrm{d}x}F_X(x) = P_X(x_1)\delta(x - x_1) + P_X(x_2)\delta(x - x_2) + g(x)$$

# Mixed RVs

In general a mixed Random variable has the generalized PDF

$$f_X(x) = \underbrace{\sum_k P(X = x_k)\delta(x - x_k)}_{\text{discrete part}} + \underbrace{g(x)}_{\text{continuous part}}$$

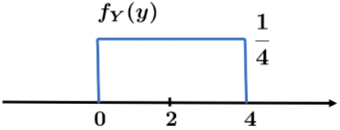
Note that we still require that

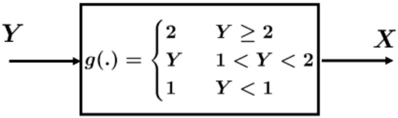
$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = \sum_k P(X = x_k) + \underbrace{\int_{-\infty}^{\infty} g(x)\mathrm{d}x}_{\leq 1} = 1$$

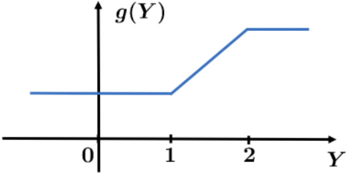
# Example

Let $Y \sim$ Uniform$(0, 4)$ and let

$$X = \begin{cases} 2 & Y \geq 2 \\ Y & 1 < Y < 2 \\ 1 & Y < 1 \end{cases}$$




- Find the CDF and (generalied) PDF of $X$.
- Find $EX$ and Var$(X)$.

# Example

Let $f_X(x) = \frac{1}{3}\delta(x+1) + \frac{1}{6}\delta(x-1) + \frac{1}{2}e^{-x}u(x)$.

1. Find $P(X=-1)$, $P(X=0)$ and $P(X=1)$
2. Find $P(X \geq 0)$
3. Find $P(X=1|X \geq 0)$
4. Find $EX$ and $\text{Var}(X)$.