APMA 1650 - Spring 2021
Lecture 16

Mon, Mar 1st, 2021

Mixed Random Variables

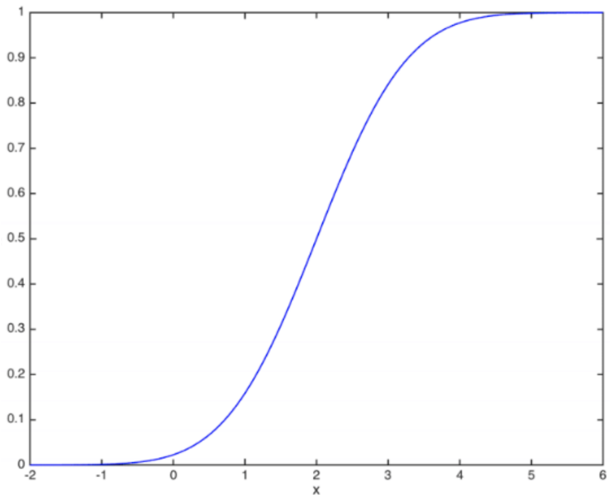
Today we will

- Introduce the generalized PDF via the delta function
- Introduce mixed random variables

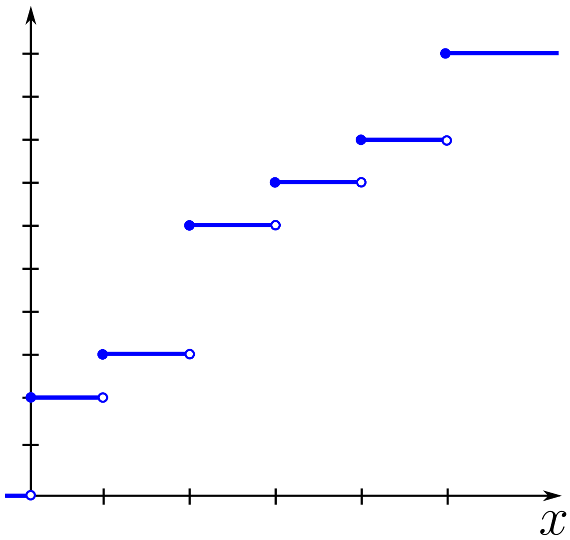
Mixed Random Variables

Recall the CDF for continuous vs discrete RV X

$$F_X(x) := P(X \leq x)$$



Continuous
Differentiable

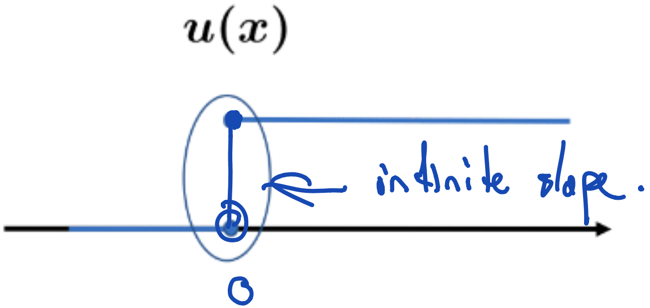


Discrete
Not Differentiable

Lets take the derivative anyway

Consider the unit step function

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



What is the derivative?

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x) = \delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

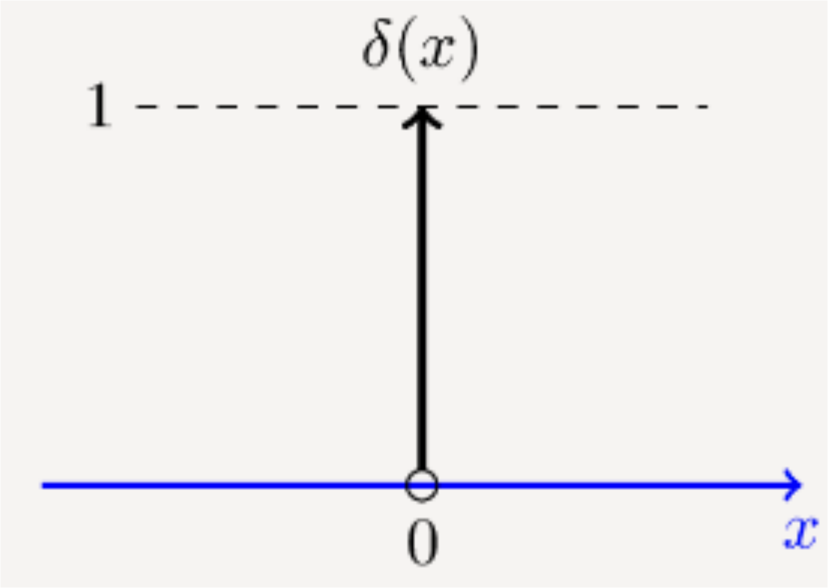
Delta “function” Not really a function.

Delta function

Note, by the "FTC" that

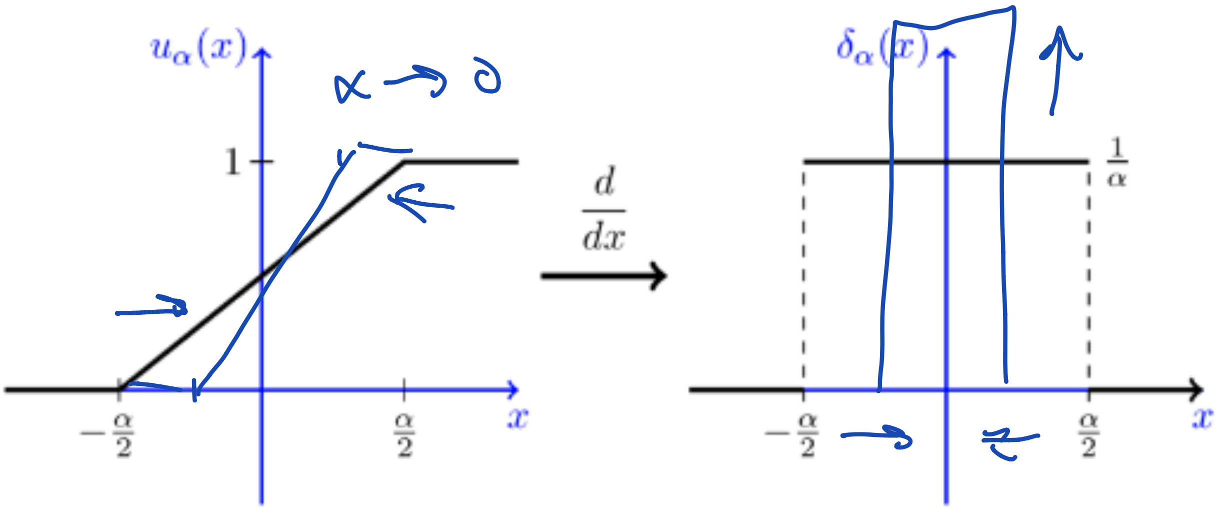
$$\text{``}\int_{-\epsilon}^{\epsilon} \delta(x)\mathrm{d}x\text{''} = \int_{-\epsilon}^{\epsilon} \frac{\mathrm{d}}{\mathrm{d}x} u(x)\mathrm{d}x = u(\epsilon) - u(-\epsilon) = 1$$

So it is like a PDF will all the probability at zero



Approximating the delta function

Best understood by approximation $u(x)$ by $u_\alpha(x)$



$$\delta_\alpha(x) = \frac{\mathrm{d}u_\alpha(x)}{\mathrm{d}x}, \quad \delta_\alpha(x) \to_{\alpha\to 0} \delta(x)$$

$$\int_{-\alpha/2}^{\alpha/2} \delta_\alpha(x)\mathrm{d}x = 1$$

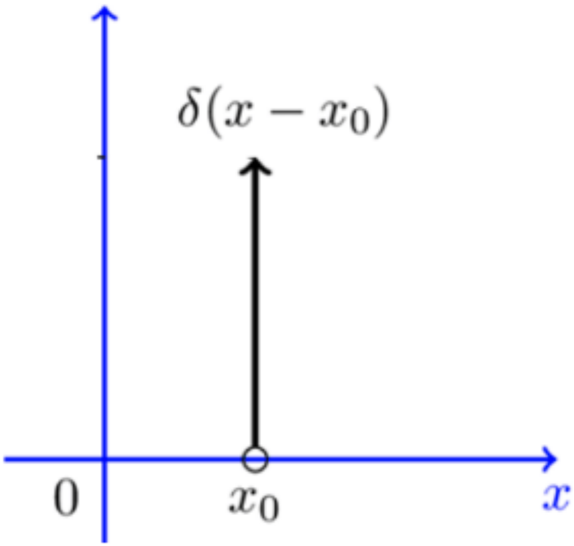
Moving the delta function around

We can shift the location of the delta function by shifting u

$$u(x - x_0) = \begin{cases} 1 & x \geq x_0 \\ 0 & \text{otherwise} \end{cases}$$

then $\delta(x - x_0) = \frac{\mathrm{d}}{\mathrm{d}x} u(x - x_0)$

$$\delta(x - x_0) = \begin{cases} \infty & x = x_0 \\ 0 & \text{otherwise} \end{cases}$$



The most important property

The delta "function" has an important property that makes it very useful

The fundamental property of delta functions

Let g be a continuous function, then

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)\mathrm{d}x = g(x_0).$$

Can replace the integral above with

$$\int_{x_0-\epsilon}^{x_0+\epsilon} g(x)\delta(x - x_0)\mathrm{d}x$$

Example

Let $g(x) = 2(x^2 + x)$, what is $\int_{-\infty}^{\infty} g(x)\delta(x-1)\mathrm{d}x$?

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-1)\,dx = g(1)$$

$$= 2(1^2+1) = 4.$$

Summary

The delta "function" has the following properties

1. $$\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x} u(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. For any continuous $g(x)$, $x_0 \in \mathbb{R}$ and $\epsilon > 0$.

$$\int_{x_0-\epsilon}^{x_0+\epsilon} g(x)\delta(x - x_0)\mathrm{d}x = g(x_0).$$

Note that choosing $g(x) = 1$ implies that

$$\int_{-\infty}^{\infty} \delta(x - x_k)\mathrm{d}x = 1.$$

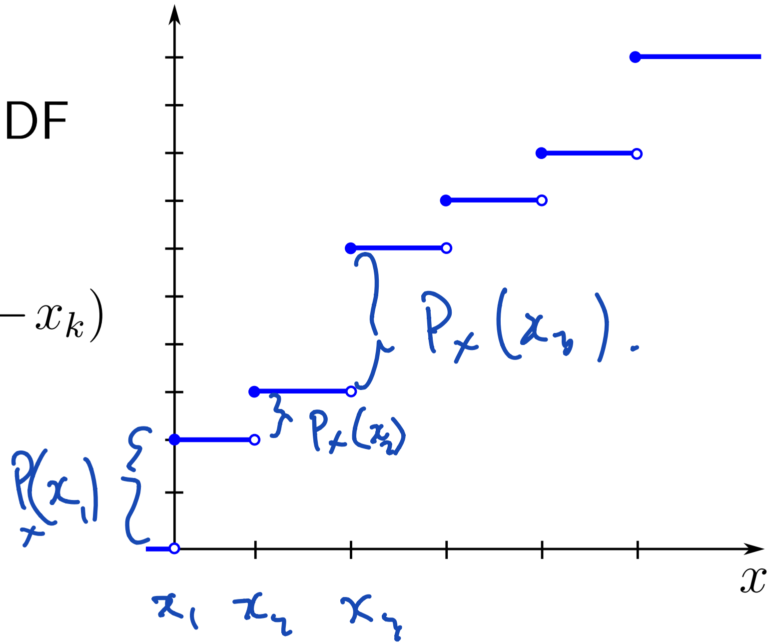
Back to discrete random variables

Consider a discrete RV X with range $R_X = \{x_1, x_2, x_3, \ldots\}$

We can always write it's CDF as a bunch of jumps

$$F_X(x) = \sum_{x_k \in R_X} P(x_k) u(x - x_k)$$

Why is this?



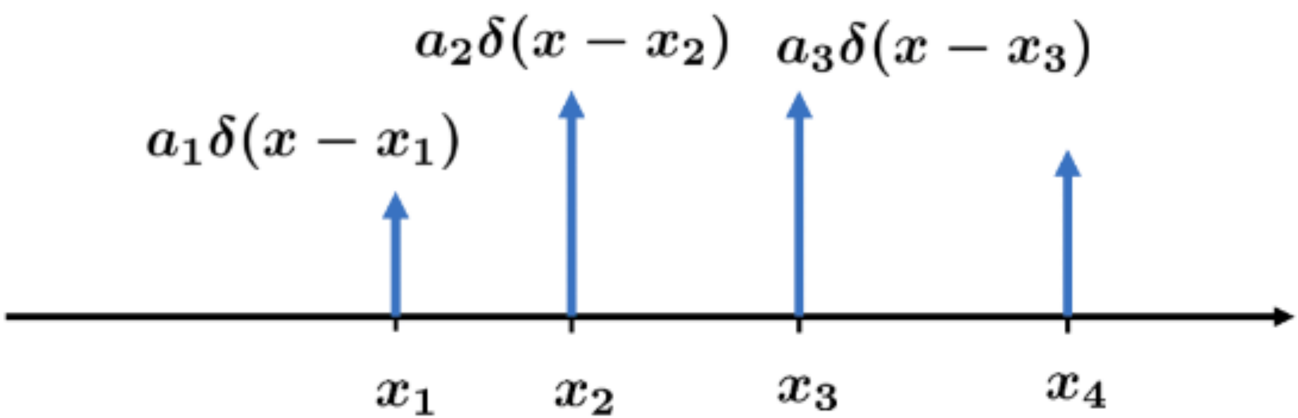
Generalized PDF

Lets take the derivative of that CDF

a_k

$$
\begin{aligned}
f_X(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{x_k \in R_X} P_X(x_k) u(x - x_k) \\
&= \sum_{x_k \in R_X} P_X(x_k) \delta(x - x_k)
\end{aligned}
$$

$\frac{d}{dx}$



Summary

Let X be a discrete RV, then its generalized PDF is given by

$$f_X(x) = \sum_{x_k \in R_X} P_X(x_k)\delta(x - x_k).$$

Can you show that this is a valid PDF?

$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1?$$

Valid PDF?

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \sum_{x_k \in R_X} P_X(x_k)\,\delta(x - x_k)\,dx$$

$$= \sum_{x_k \in R_X} P_X(x_k) \underbrace{\int_{-\infty}^{\infty} \delta(x - x_k)\,dx}_{=1}$$

$$= \sum_{x_k \in R_X} P_X(x_k) = 1.$$

Expectation

What happens with the expectation

$$EX = \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x?$$

$$EX = \int_{-\infty}^{\infty} x \left(\sum_{x_k \in R_X} P_X(x_k)\, \delta(x - x_k) \right) dx$$

$$= \sum_{x_k \in R_X} P_X(x_k) \underbrace{\int_{-\infty}^{\infty} x\, \delta(x - x_k)\, dx}_{= x_k}$$

$$\boxed{= \sum_{x_k \in R_X} x_k P_X(x_k).}$$

Example

From the HW

$$F_X(x) = \begin{cases} 0 & x < -2 \\ 1/3 & -2 \leq x < -1 \\ 1/2 & -1 \leq x < 1 \\ 5/6 & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$

$5/6 - 1/2 = 1/3$

What is $f_X(x)$?

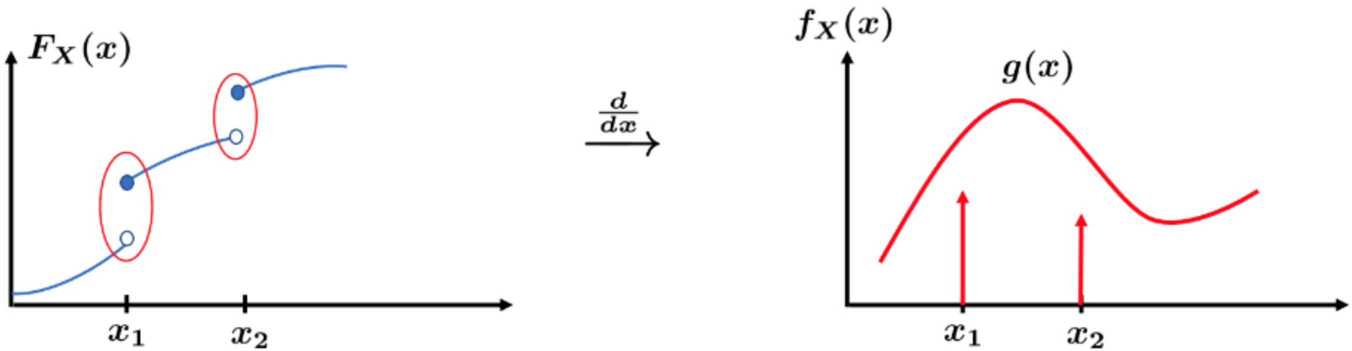
$x+2$

$f_X(x) = 1/3\, \delta(x-(-2)) + 1/6\, \delta(x+1)$

$+ 1/3\, \delta(x-1) + 1/6\, \delta(x-2)$

Mixed Random variables

In general a Random variable can be a mix of discrete and continuous parts



$$\frac{\mathrm{d}}{\mathrm{d}x}F_X(x) = P_X(x_1)\delta(x - x_1) + P_X(x_2)\delta(x - x_2) + g(x)$$

actual function.

Mixed RVs

In general a mixed Random variable has the generalized PDF

a_k

$$f_X(x) = \underbrace{\sum_k P(X = x_k)\delta(x - x_k)}_{\text{discrete part}} + \underbrace{g(x)}_{\text{continuous part}}$$

Note that we still require that

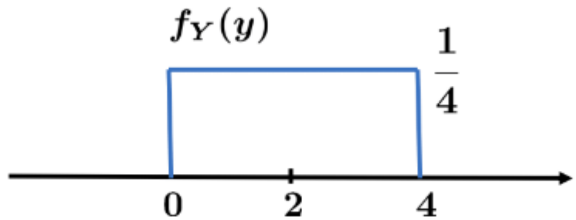
$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = \sum_k P(X = x_k) + \underbrace{\int_{-\infty}^{\infty} g(x)\mathrm{d}x}_{\leq 1} = 1$$

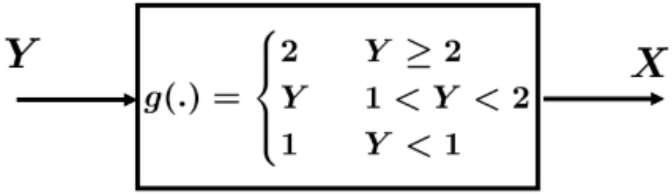
Example

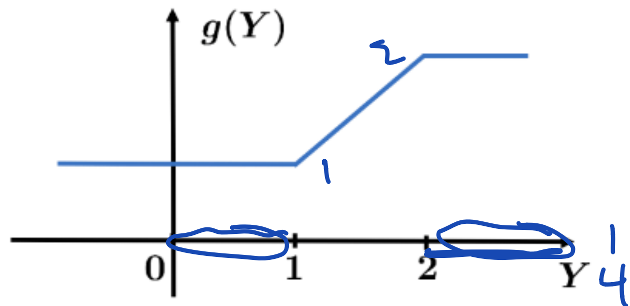
$P(0 < Y \leq 1)$

Let $Y \sim$ Uniform$(0, 4)$ and let

$$X = \begin{cases} 2 & Y \geq 2 \\ Y & 1 < Y < 2 \\ 1 & Y < 1 \end{cases}$$







- Find the CDF and (generalied) PDF of X.
- Find EX and Var(X).

$$f_X(x) = a_1\,\delta(x-1) + a_2\,\delta(x-2) + g(x)$$

$$a_1 = P(X=1) = P(Y \in [0,1])$$

$$a_2 = P(Y \in [2,4])$$

$$a_1 = \int_0^1 \tfrac{1}{4}\,dx = \tfrac{1}{4}$$

$$a_2 = P(Y \in [2,4]) = \int_2^4 \tfrac{1}{4}\,dx = \frac{4-2}{4} = \tfrac{1}{2}$$

$$f_X(x) = \tfrac{1}{4}\,\delta(x-1) + \tfrac{1}{2}\,\delta(x-2) + g(x)$$

$$g(x) = \begin{cases} c & x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

How to find c:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_{-\infty}^{\infty} \tfrac{1}{4}\delta(x-1) + \tfrac{1}{2}\delta(x-2) + c\int_1^2 1\,dx.$$

$$= \tfrac{1}{4} + \tfrac{1}{2} + c = 1 \Rightarrow c = 1 - \tfrac{1}{4} - \tfrac{1}{2} = \tfrac{1}{4}$$

$$f_X(x) = \tfrac{1}{4}\delta(x-1) + \tfrac{1}{2}\delta(x-2) + \begin{cases} \tfrac{1}{4} & x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

Example

Let $f_X(x) = \frac{1}{3}\delta(x+1) + \frac{1}{6}\delta(x-1) + \frac{1}{2}e^{-x}u(x)$.

1. Find $P(X \overset{``1/3}{=} -1)$, $P(X = 0)$ and $P(X \overset{``1/6}{=} 1)$

   $``0$ ← no delta function at zero.

2. Find $P(X \geq 0)$
3. Find $P(X = 1 | X \geq 0)$
4. Find EX and $\mathrm{Var}(X)$.

2. $$P(X \geq 0) = \int_0^\infty f_X(x)\,dx$$

$$= \underbrace{\int_0^\infty \tfrac{1}{3}\delta(x+1)\,dx}_{=0} + \underbrace{\int_0^\infty \tfrac{1}{6}\delta(x-1)}_{=1/6} + \underbrace{\int_0^\infty \tfrac{1}{2}e^{-x}\,dx}_{=1/2}$$

$$= 1/6 + 1/2 = \frac{2}{3}$$

3) $$P(X=1 \mid X \geq 0) = \frac{P(X=1 \text{ and } X \geq 0)}{P(X \geq 0)}$$

$$= \frac{P(X=1)}{P(X \geq 0)} = \frac{1/6}{2/3} = 1/4$$

4) $$EX = \int_{-\infty}^{\infty} x\left(1/3\, \delta(x+1) + 1/6\, \delta(x-1) + 1/2\, e^{-x} u(x)\right) dx$$

$$= 1/3(-1) + 1/6(1) + 1/2 \underbrace{\int_0^{\infty} x e^{-x}\, dx}_{\text{"}1}$$

"Integration by parts".

$$= -1/3 + 1/6 + 1/2$$

$$= 1/3$$

4) $\mathrm{Var}(X) = E(X^2) - (EX)^2$

$$EX^2 = \int_{-\infty}^{\infty} x^2 \left(\tfrac{1}{3}\delta(x+1) + \tfrac{1}{6}\delta(x-1) + \tfrac{1}{2} e^{-x} u(x) \right) dx$$

$$= \tfrac{1}{3}(-1)^2 + \tfrac{1}{6}(1)^2 + \tfrac{1}{2}\int_0^{\infty} x^2 e^{-x}\, dx$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\int_0^{\infty} x^2 e^{-x}\, dx = \tfrac{1}{2} + 1 = \tfrac{3}{2}$$

FBP.

$$\boxed{\begin{aligned} \int_0^{\infty} x^2 e^{-x}\, dx &= \underbrace{-x^2 e^{-x}\Big|_0^{\infty}}_{=0} + 2\int_0^{\infty} x e^{-x}\, dx \\ &= \underbrace{-2x e^{-x}\Big|_0^{\infty}}_{=0} + 2\int_0^{\infty} e^{-x}\, dx = 2. \end{aligned}}$$

$$\mathrm{Var}(X) = \tfrac{3}{2} - \left(\tfrac{1}{3}\right)^2 = \tfrac{3}{2} - \tfrac{1}{9} = \tfrac{25}{18}.$$