Announcements

- Second midterm exam will be moved to W 3/31. (Week 11).

Lecture 17 - A primer on multivariable calc.

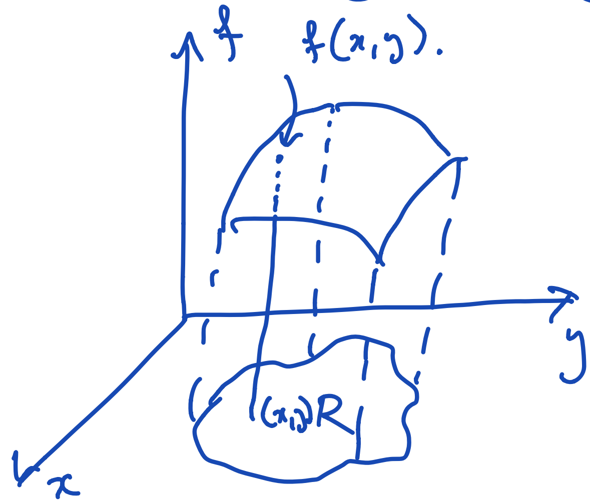
We will be interested in functions of more than one variable.

$$f(x, y)$$

both take values in $\mathbb{R}$

$$f : \underbrace{\mathbb{R} \times \mathbb{R}}_{\mathbb{R}^2} \longrightarrow \mathbb{R}$$

Visualize this by graphing a surface.



Partial derivative.

$$\partial_x f(x,y) = \frac{\partial}{\partial x} f(x,y) = f_x(x,y)$$
$$\equiv \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\partial_y f(x,y) = \frac{\partial}{\partial y} f(x,y) = f_y(x,y)$$
$$\equiv \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

This is just like freezing one variable and taking the (single variable) derivative with respect to the other.

Allows to define. $\partial_y^2 f(x,y) = \frac{\partial^2}{\partial y^2} f(x,y).$

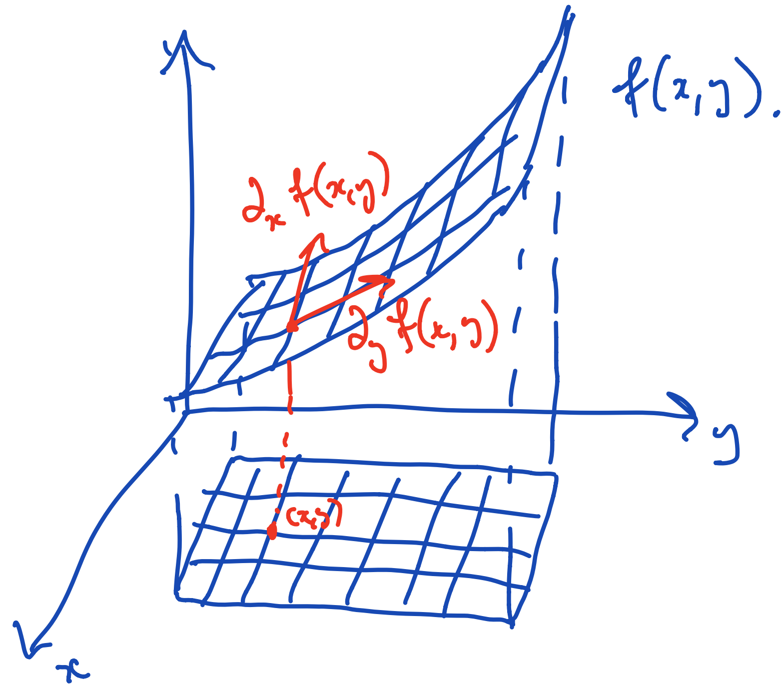
$$\partial_x^2 f(x,y) = \frac{\partial^2}{\partial x^2} f(x,y).$$

Mixed derivatives $\partial_x \partial_y f(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y).$

if f is "nice" then the order doesn't matter.

$$\boxed{\partial_x \partial_y f(x,y) = \partial_y \partial_x f(x,y).}$$

Picture



Example $f(x,y) = 2x^2y^2$

$\partial_x f \quad , \partial_y f \quad , \quad \partial_x \partial_y f \quad ?$

$$\partial_x f(x,y) = \partial_x (2x^2 \underset{\uparrow \text{ const.}}{y^2}) = 2y^2 \partial_x x^2 = 4y^2 x.$$

$$\partial_y f(x,y) = \partial_y (2\underset{\uparrow \text{ const}}{x^2} y^2) = 4x^2 y.$$

$$\partial_x \partial_y f(x,y) = \partial_x (4x^2 y) = 8xy.$$

$$\partial_y \partial_x f(x,y) = \partial_y (4y^2 x) = 8xy.$$

same.

Partial Integrals

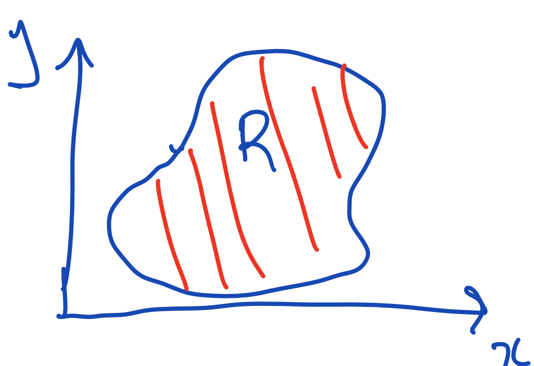
Can also integrate in each direction.

$$\int_a^b f(x, \underset{\substack{\uparrow \\ \text{const}}}{y})\, dx \qquad \text{or} \qquad \int_c^d f(\underset{\substack{\uparrow \\ \text{const.}}}{x}, y)\, dy$$

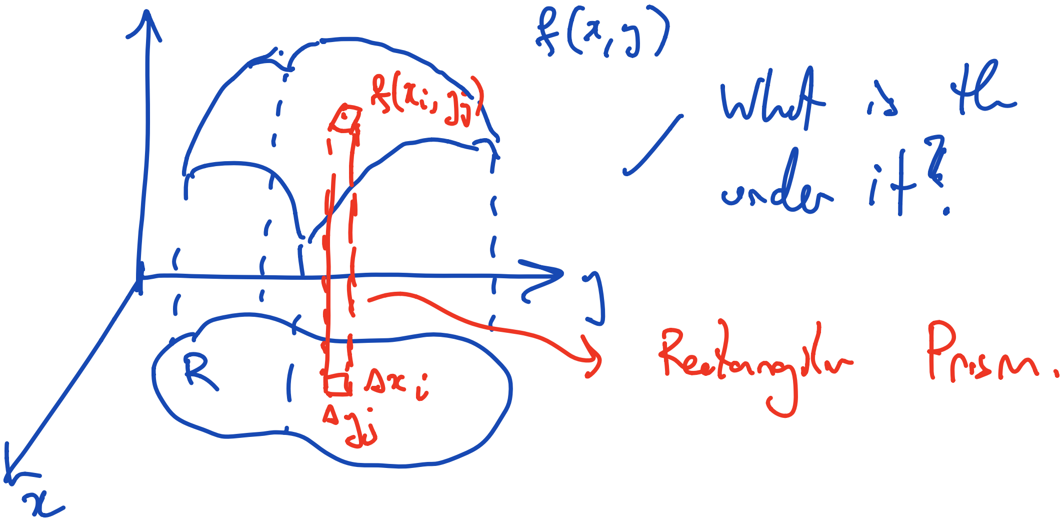
FTC.

- $\displaystyle\int_a^b \partial_x f(x,y)\, dx = f(b,y) - f(a,y)$
- $\displaystyle\int_c^d \partial_y f(x,y)\, dy = f(x,d) - f(x,c).$
- $\displaystyle\int_a^b \partial_y f(x,y)\, dx = \frac{d}{dy}\left(\int_a^b f(x,y)\, dx\right)$

Area integrals



Given $f(x,y)$. What is is the area under f?
(For $(x,y) \in R$).



What is the volume under it?

$$\text{Volume} \simeq \sum_{i,j} f(x_i, y_j)\, \Delta x_i \Delta x_j$$

Take the limit.

$$\boxed{\text{Volume} = \iint_R f(x,y)\, dA.}$$

↑ area element.

How to compute this??

This strongly depends on R.

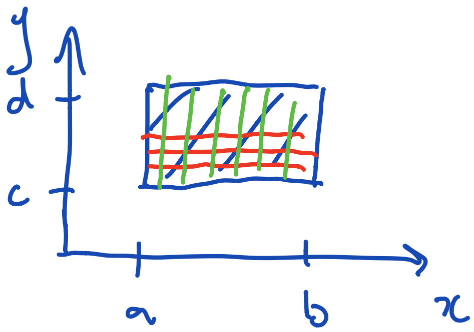
$\Rightarrow$ iterated partial integrals..

3 situations.

1. Rectangular Region.
2. Region Bounded between two curves.
3. Polar Regions (polar coordinates).

Rectangular Coordinates.

① If R is a rectangle. $R = [a,b] \times [c,d]$.



$$R = \{(x,y) : a \le x \le b, \ c \le y \le d\}.$$

"$dA = dx\,dy$".

area element in rectangular coordinates.

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Example

$$\iint_R 6xy^2\,dA \qquad R = [2,4]\times[1,2].$$

$$= \int_2^4 \left(\int_1^2 6xy^2\,dy \right) dx = \int_2^4 6x \left(\int_1^2 y^2\,dy \right) dx$$

$$= \left(\int_1^2 y^2\,dy \right) \left(\int_2^4 6x\,dx \right)$$

$$= \left(\tfrac{1}{3} y^3 \Big|_1^2 \right) \left(\tfrac{6}{2} x^2 \Big|_2^4 \right) = \left(\frac{8}{3} - \frac{1}{3} \right) \left(3(16-4) \right)$$

$$= \left(\frac{7}{\cancel{3}} \right) \left(\cancel{3}(12) \right)$$

$$= 7(12) = 84.$$

Ex $$\iint_R x e^{xy}\,dA \qquad R = [-1,2]\times[0,1].$$

Try to integrate - first in y then x.

- first in x then y.

We will continue this next class.