Primer on multivariable continued.

Recall From last lecture.

Ex $\iint_R x e^{xy}\, dA \qquad R : [-1, 2] \times [0, 1].$

- order of integration can make or break the calculation.

$$= \int_{-1}^{2} \left(\int_0^1 x e^{xy}\, dy \right) dx = \int_{-1}^{2} \left(\int_0^{x} e^{u}\, du \right) dx$$

$u = xy \quad du = x\,dy \qquad u = x(0) = 0 \qquad u = x(1) = x$

$$= \int_{-1}^{2} \left(e^{x} - 1 \right) dx = e^{x} - x \Big|_{-1}^{2}$$

$$= e^{2} - e^{-1} - 3$$

$$\int_0^1 \left(\int_{-1}^{2} x e^{xy}\, dx \right) dy = \int_0^1 \left(\frac{x}{y} e^{xy} \Big|_{-1}^{2} - \int_{-1}^{2} \frac{1}{y} e^{xy} dx \right) dy$$

x

$u = x \qquad dv = e^{xy} dx$

$du = dx \qquad v = \frac{1}{y} e^{xy}$

$$= \int_0^1 \left(\frac{2}{y} e^{2y} + \frac{1}{y} e^{-y} - \frac{1}{y} \int_{-1}^{2} e^{xy}\, dx \right) dy$$

$$= \int_0^1 \frac{2}{y} e^{2y} + \frac{1}{y} e^{-y} - \frac{1}{y^2} e^{2y} + \frac{1}{y^2} e^{-y}\, dy .$$

$$\int \frac{e^{y}}{y}\, dy = ?? \quad = Ei(y)$$ "Exponential integral"

special functions.

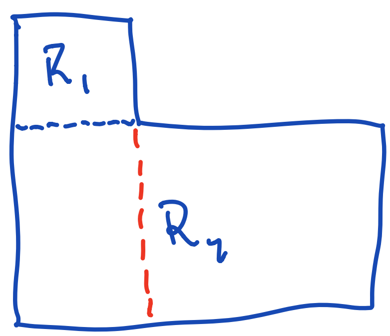
More general Regions.

Additive property (behaves like a measure).

$R = R_1 \cup R_2$

↑ disjoint

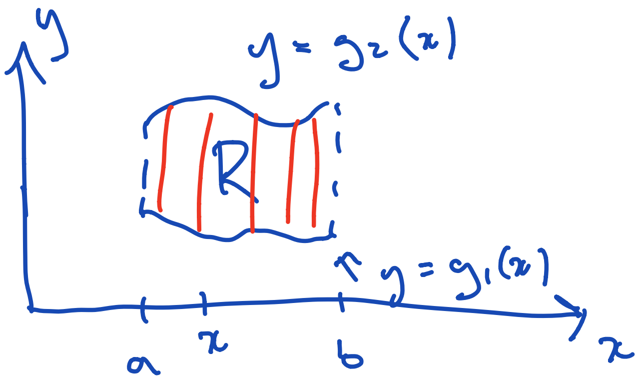
Ex



$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA .$$

Regions Bounded between two curves

- Two horizontal curves.



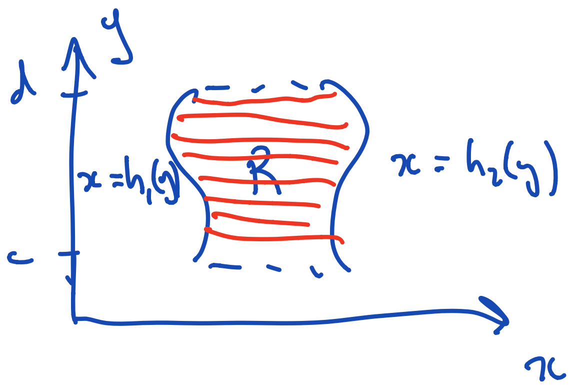
$R = \{(x,y) : a \le x \le b$

$g_1(x) \le y \le g_2(x)\}.$

"$dA = dx\,dy$".

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx.$$

- Two vertical curves.

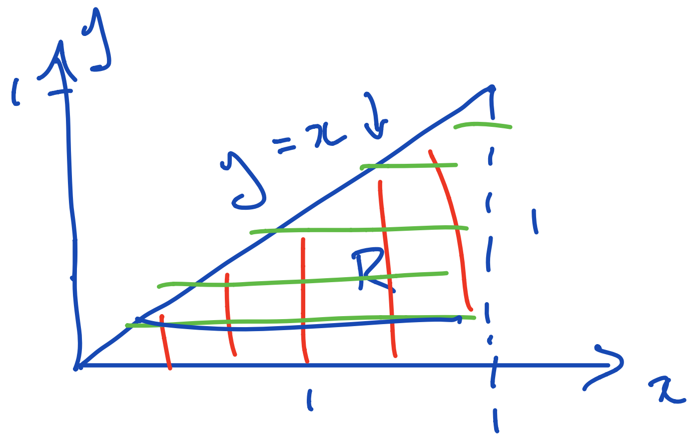


$R = \{(x,y) : c \le y \le d$

$h_1(y) \le x \le h_2(y)\}.$

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy.$$

Example Triangle



$$R = \{(x,y) : 0 \le x \le 1$$
$$0 \le y \le x\}$$

$g_1(x)$ $\quad$ $g_2(x)$.

$$\iint_R 6xy^2 \, dA = \int_0^1 \left(\int_0^x 6xy^2 \, dy \right) dx$$

$$= \int_0^1 \left(\frac{6x}{3} y^3 \Big|_0^x \, dy \right.$$

$$= \int_0^1 2x^4 \, dx = \frac{2}{5} x^5 \Big|_0^1 = \frac{2}{5}.$$

Horizontal integration. $R = \{(x,y) : 0 \le y \le 1$
$$y \le x \le 1\}$$

$$\iint_R 6xy^2 \, dA = \int_0^1 \left(\int_y^1 6xy^2 \, dx \right) dy.$$

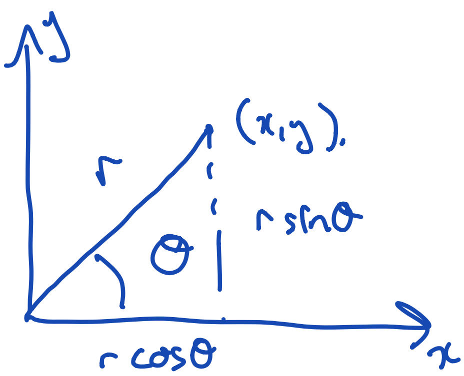
$$= \int^1 \left(\frac{6}{\;} x^2 y^2 \Big|^1 \, dy \right.$$

$$= \int_0^1 3y^2 - 3y^4 \, dy = y^3 - \frac{3}{5} y^5 \Big|_0^1$$

$$= 1 - 3/5 = 2/5$$

Polar Coordinates.

$$\underset{\text{Rectangular.}}{(x, y)} \longmapsto \underset{\text{Polar.}}{(r, \theta)}$$

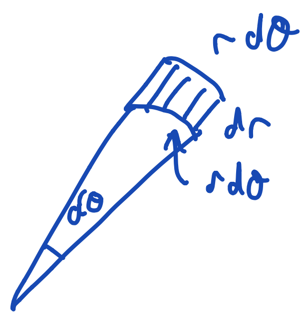


$$r = \sqrt{x^2 + y^2}$$

$$\theta = \text{"}\operatorname{atan2}\text{"}(x, y) \in [0, 2\pi].$$

$$= \begin{cases} \arctan(y/x) & x \geq 0 \\ \arctan(y/x) + \pi & x \leq 0 \quad y \geq 0 \\ \arctan(y/x) - \pi & x \leq 0 \quad y < 0 \end{cases}$$

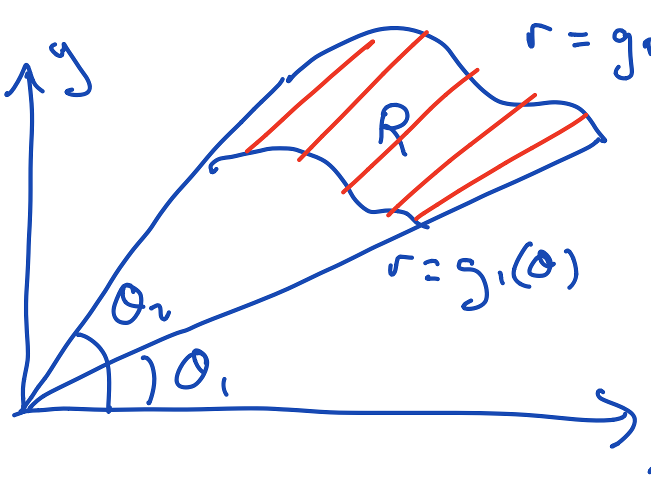
$$dA = r \, dr \, d\theta$$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

General Polar region.



$$R = \{(r\cos\theta, r\sin\theta) \; \theta_1 \le \theta \le \theta_2 \; g_1(\theta) \le r \le g_2(\theta)\}.$$

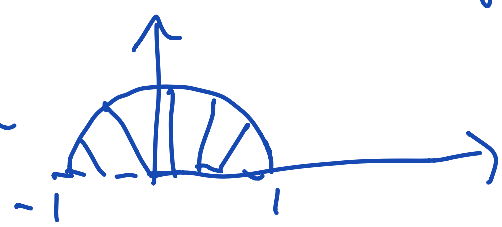
$$\iint_R f(x,y)\, dA = \int_{\theta_1}^{\theta_2} \left(\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr \right) d\theta.$$

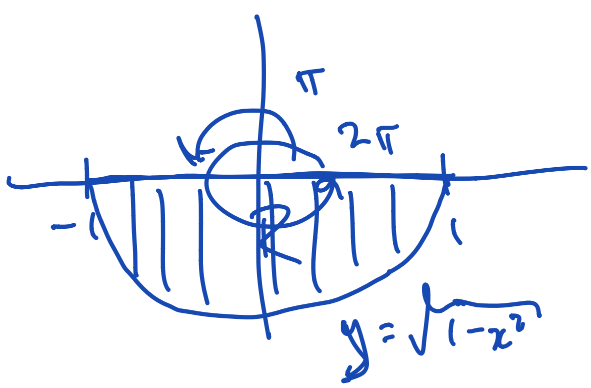
Example

$$y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$$

$$y = \sqrt{1-x^2} \qquad r^2 = 1 \qquad r = 1$$

$$\int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{0} \cos(x^2+y^2)\, dy \right) dx$$





$$R = \{(r\cos\theta, r\sin\theta) : \; \pi \le \theta \le 2\pi \; 0 \le r \le 1\}.$$

$$\int_{\pi}^{2\pi} \left(\int_0^1 \cos(r^2)\, r\, dr \right) d\theta$$

$u = r^2$

$du = 2r\,dr$

$$= \int_{\pi}^{2\pi} \left(\int_0^1 \tfrac{1}{2} \cos(u)\, du \right) d\theta = \int_{\pi}^{2\pi} \tfrac{1}{2} \sin(u) \Big|_0^1 d\theta$$

$$= \int_{\pi}^{2\pi} \tfrac{1}{2} \sin(1)\, d\theta = \tfrac{\pi}{2} \sin(1).$$

Example. $$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\pi/a}$$

$$\left(\int_{-\infty}^{\infty} e^{-ax^2}\, dx \right)^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2}\, dx \right) \left(\int_{-\infty}^{\infty} e^{-ay^2}\, dy \right)$$

Double the variables. $\rightarrow$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ax^2} e^{-ay^2}\, dx\, dy.$$

$$= \iint_{\mathbb{R}^2} e^{-a(x^2+y^2)}\, dA.$$

Polar Coordinates

$$= \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r\, dr\, d\theta.$$

$u = r^2$

$u = r^2$

$du = 2r\,dr.$

$$= \int_0^{2\pi} \int_0^{\infty} \tfrac{1}{2} e^{-au}\, du\, d\theta = -\frac{2\pi}{2a} e^{-au} \Big|_0^{\infty}$$

$$= \pi / a.$$

Therefore $\left(\int_{-\infty}^{\infty} e^{-ax^2} dx \right)^2 = \pi / a.$