Joint Probability. (Multivariate Probability).

- Dealing with 2 (or more) RVs
- How are two RVs related?

Joint Probability. (Discrete).

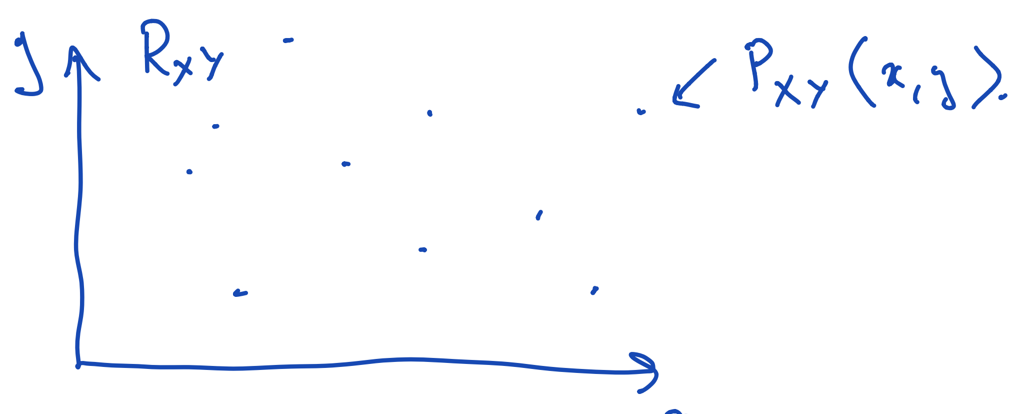
Let X, Y be two discrete RVs then

$$P_{XY}(x,y) = P(X = x \underset{\text{and}}{,} Y = y)$$

is the joint PMF of X and Y.

with range.

$$R_{XY} = \{(x,y) : x \in R_X, y \in R_Y, P_{XY}(x,y) \neq 0\}$$



Properties

① $0 \le P_{XY}(x,y) \le 1$

② $$\sum_{(x,y)\in R_{XY}} P_{XY}(x,y) = 1$$

③ $A \subset \mathbb{R}^2$ $\quad P((X,Y)\in A) = \sum_{(x,y)\in A\cap R_{XY}} P_{XY}(x,y).$

Ex Roll two dice.. $\quad X =$ outcome 1

$Y =$ outcome 2.

X\Y	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	⋮	⋮				
3	⋮	⋮		1/36	1/36	1/36
4	⋮	⋮				
5	⋮	⋮	1/36			
6	1/36	1/36	…	…	…	1/36

Suppose $Z = 7 - X$

$X \backslash Z$	1	2	3	4	5	6
1	0	0	0	0	0	1/6
2	0	0	0	0	1/6	0
3	0	0	0	1/6	0	0
4	0	0	1/6	0	0	0
5	0	1/6	0	0	0	0
6	1/6	0	0	0	0	0

<u>Marginals</u> Let X, Y be two discrete RVs with joint PMF $P_{XY}(x, y)$.

How are X and X distributed?

<u>Ex</u>. $P(X = 3)$?

Law of total probability.

$$P(X=3) = \sum_{y \in R_Y} P(X = 3, Y = y)$$

$$= \sum P_{XY}(3, y)$$

$y \in R_Y$

Definition X, Y discrete PMF $P_{XY}(x,y)$.

the marginals of X, Y are:

$$P_X(x) = \sum_{y \in R_Y} P_{XY}(x,y)$$

$$P_Y(y) = \sum_{x \in R_X} P_{XY}(x,y).$$

Example

$X \backslash Y$	0	1	P_X
0	$1/2$	$1/3$	$1/2 + 1/3 = 5/6$
1	$1/6$	0	$1/6$
P_Y	$2/3$	$1/3$	1

Are X, Y independent?

$$P(Y=0 \mid X=0) = \frac{P(Y=0, X=0)}{P(X=0)}$$

$$= \frac{p_{XY}(0,0)}{p_X(0)} = \frac{1/2}{5/6}$$

$$= 3/5$$

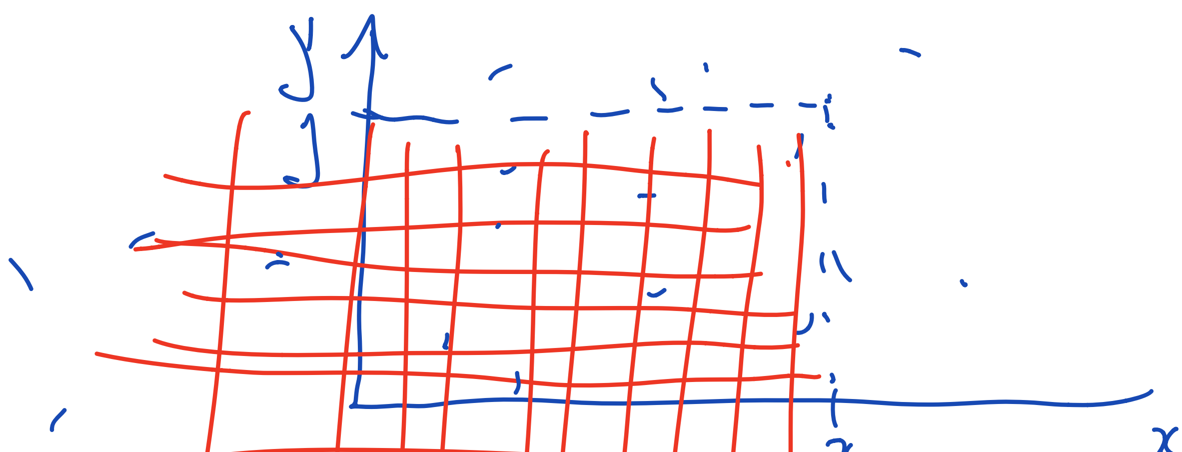
$$\neq 2/3 = P(Y=0).$$

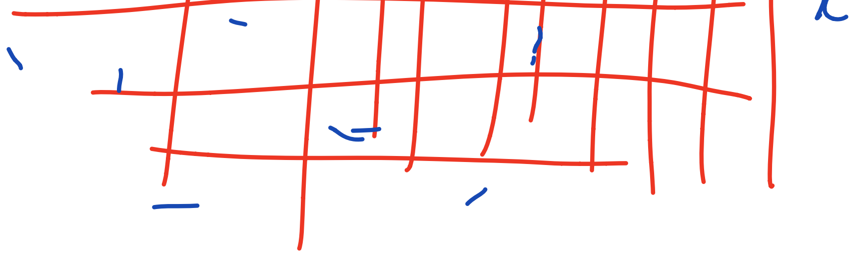
They are not independent!!

Joint CDF

X, Y are discrete.

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$





<u>Properties.</u>

① $0 \le F_{XY}(x,y) \le 1$

② Non-decreasing in each variable.

③ $F_X(x) = F_{XY}(x,\infty) = P(X \le x, Y \cancel{<} \infty)$

$F_Y(y) = F_{XY}(\infty, y) = P(X \cancel{<} \infty, Y \le y)$

④ $F_{XY}(\infty, \infty) = 1$

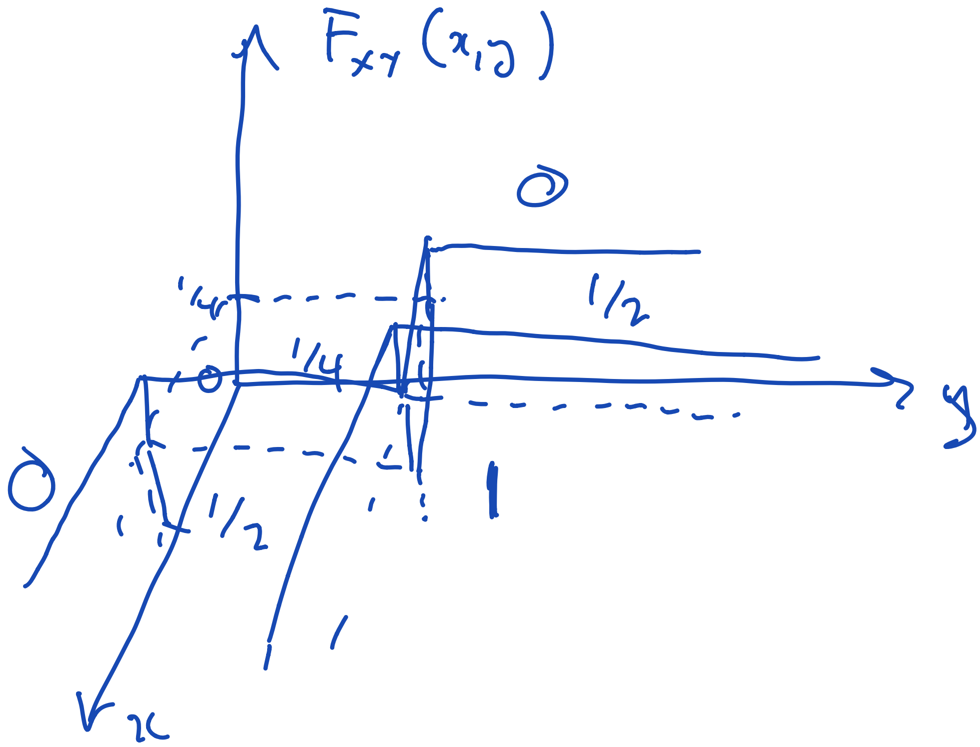
$F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$

<u>Ex</u> Toss a coin twice.

$X = \begin{cases} 1 & H \\ 0 & T \end{cases} \uparrow$ first $\quad Y = \begin{cases} 1 & H \\ 0 & T \end{cases} \uparrow$ second.

F_{XY}

$X \backslash Y$	0	1
0	1/4	1/2
1	1/2	1



Independence

Recall X, Y are independent if and only if

$$P(X \le x, Y \le y) = P(X \le x)\, P(Y \le y)$$

or $\Updownarrow$

$$F_{XY}(x,y) = F_X(x)\, F_Y(y).$$

also

$$P_{XY}(x,y) = P_X(x)\,P_Y(y)$$

Ex

$X \backslash Y$	1	2	P_X
1	$1/3$	$1/12$	$5/12$
2	$1/6$	0	$1/6$
4	$1/12$	$1/3$	$5/12$
P_Y	$7/12$	$5/12$	1

$$P(X=1 \mid Y=2) = \frac{P_{XY}(1,2)}{P_Y(2)} = \frac{1/12}{5/12}$$

$$= 1/5$$

$$\neq 5/12$$

$$P_X(1).$$

Not independent.