# Conditional Distributions

Recall, $X, Y$ are independent if

$$P(X \leq x, Y \leq x) = P(X \leq x) P(Y \leq x)$$

or

$$F_{XY}(x,y) = F_X(x) F_Y(y).$$

Discrete | Continuous

$$P_{XY}(x,y) = P_X(x) P_Y(y) \qquad f_{XY}(x,y) = f_X(x) f_Y(y)$$

What if they are not independent?

## Conditioning

Recall: $A, B,$ $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}.$

Def: $X$ be a discrete RV., $A$ event.

$$P(X = x \mid A) = \frac{P(\{X = x\} \cap A)}{P(A)}.$$

(PMF)

Or the conditional distribution of "$X|A$."

$$P_{X|A}(x) = \frac{P(X=x \overset{\text{and.}}{,} A)}{P(A)}.$$

Particularly useful when there are two RVs. $X, Y$ (discrete). Definition

"$A = \{Y = y\}$."

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}.$$

Conditional distribution of $X$ wrt $Y$.

"$P_{X|Y}(x|y) = P(X=x \mid Y=y)$."

Ex

| $X \backslash Y$ | 1 | 2 | $P_X$ |
|---|---|---|---|
| 1 | 1/3 | 1/12 | 5/12 |
| 2 | 1/6 | 0 | 1/6 |
| 4 | 1/12 | 1/3 | 5/12 |
| $P_Y$ | 7/12 | 5/12 | 1 |

$P_{XY}(x,y)$.

What is $P_{X|Y}(x|2)$

$$P_{X|Y}(x|2) = \frac{P_{XY}(x,2)}{5/12}$$

| $x$ | 1 | 2 | 4 | |
|---|---|---|---|---|
| $P_{X\|Y}(x\|2)$ | $1/5$ | $0$ | $4/5$ | $= 1$ |

New distribution.

$$= \frac{1/12}{5/12} \qquad = \frac{0}{5/12} \qquad = \frac{1/3}{5/12}$$

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## Conditional Expectation + Variance.

Recall

$$\mathbb{E} X = \sum_{x \in R_X} x P_X(x).$$

**Def** $X, Y$ discrete the the conditional expectation is.

$$\mu_{X|Y}(y) = \mathbb{E}\{X \mid Y = y\} = \sum_{x \in R_X} x \, P_{X|Y}(x|y)$$

Also

$$\mathrm{Var}(X \mid Y = y) = \sum_{x \in R_X} (x - \mu_{X|Y}(y))^2 P_{X|Y}(x|y).$$

**Example**

| $x \backslash y$ | 1 | 2 | $P_X$ |
|---|---|---|---|
| 1 | $1/3$ | $1/12$ | $5/12$ |
| 2 | $1/6$ | $0$ | $1/6$ |
| 4 | $1/12$ | $1/3$ | $5/12$ |
| $P_Y$ | $7/12$ | $5/12$ | $1$ |

$$\mathbb{E}\{X \mid Y = 2\} = \sum_{x \in R_X} x \, P_{X|Y}(x \mid 2)$$

$$= 1\left(1/5\right) + 2\left(0\right) + 4\left(4/5\right)$$

$$= 1/5 + \frac{16}{5} = 17/5$$

$$\mathrm{Var}(X \mid Y = 2) = \sum_{x \in R_X} \left(x - 17/5\right)^2 P_{X|Y}(x \mid 2)$$

$$= \left(1 - \frac{17}{5}\right)^2 \left(1/5\right) + \left(2 - \frac{17}{5}\right)^2 \left(0\right) + \left(4 - \frac{17}{5}\right)^2 \left(4/5\right)$$

$$= \left(\frac{12}{5}\right)^2 \left(1/5\right) + \left(\frac{3}{5}\right)^2 \left(\frac{4}{5}\right) = \frac{36}{25}$$

## Law of total probability.

$$P_X(x) = \sum_{y \in R_Y} P_{XY}(x, y)$$

$$= \sum_{y \in R_Y} P_{X|Y}(x|y)\, P_Y(y).$$

Often easier to compute.
than $P_{XY}$

Law of total Expectation.

$$EX = \sum_{y \in R_Y} \mathbb{E}[X \mid Y = y]\, P_Y(y).$$

## Law of iterated Expectation.

$\mu_{X|Y}(y) = \mathbb{E}[X \mid Y = y]$ - function of $R_Y$

Define

$$\mathbb{E}[X \mid Y] = \mu_{X|Y}(Y)$$

Is a Random Variable!! function of $Y$.

Law of iterated expectation.

"X|Y" doesn't mean anything.

$$\mathbb{E}X = \mathbb{E}\{\underbrace{\mathbb{E}\{X \mid Y\}}_{\text{Random..}}\}$$

Properties.

① $\mathbb{E}[g(X)h(Y) \mid Y] = h(Y)\,\mathbb{E}[g(X) \mid Y]$

② If $X$ and $Y$ are independent.

$$\mathbb{E}\{g(X) \mid Y\} = \mathbb{E}\{g(X)\}.$$

Ex Own a bar. On a given day. there are

$N \sim \text{Poisson}(a)$. customers,

Each customer buys a drink with prob p.

What are the expected # of customers who by a drink?

$X_i \sim \text{Bernoulli}(p)$ independent.

$$X = \sum_{i=1}^{N} X_i \quad \text{- \# of drinks bought.}$$

$N$ random

$$\mathbb{E}X = \mathbb{E}\{\mathbb{E}\{X \mid N\}\}.$$

$\mu_{X|N}(N)$

"$\mu_{X|N}(n)$" $= np$

treat $N$ like constant.

$$= \mathbb{E}\left\{\mathbb{E}\left\{\sum_{i=1}^{N} X_i \mid N\right\}\right\} = \mathbb{E}\{Np\}.$$

Binomial$(N, p)$

$$= p\,\mathbb{E}N = p\lambda.$$

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## Law of Total Variance.

$$\mathrm{Var}(X) = \mathbb{E}\{\mathrm{Var}(X \mid Y)\} + \mathrm{Var}(\mathbb{E}[X \mid Y])$$

$\sigma^2_{X|Y}(Y)$   $\mu_{X|Y}(Y)$.

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Ex $\quad X = \sum_{i=1}^{N} X_i \quad \mathrm{Var}(X).$

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X \mid N)) + \mathrm{Var}(\mathbb{E}\{X \mid N\})$$

$$\mathrm{Var}(X \mid n) = \mathrm{Var}\Big(\underbrace{\sum_{i=1}^{n} X_i}_{\text{Binomial}(n,p)} \;\Big|\; n\Big) = npq$$

$$\mathbb{E}[X \mid n] = \mathbb{E}\Big(\sum_{i=1}^{n} X_i \;\Big|\; n\Big) = np.$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}[Npq] + \mathrm{Var}(Np)$$

$$= pq\,\mathbb{E}N + p^2\,\mathrm{Var}(N)$$

$$= pq\lambda + p^2\lambda$$

$$= \lambda\big(p(1-p) + p^2\big) = \lambda p.$$