Announcements

- Can ignore (if you want.) sections 5.1.4, 5.2.4 on "functions of two RVs".

Jointly Continuous RVs.

Def: We say that two continuous RVs X, Y are jointly continuous if there exists a joint PDF

$$f_{XY}(x,y)$$

s.t.

$$P\Big(\underbrace{(X,Y)}_{\in \mathbb{R}^2} \in R\Big) = \iint_R f_{XY}(x,y)\, dA$$

Properties

① $f_{XY}(x,y) \geq 0$, $R_{(X,Y)} = \{(x,y) : f(x,y) > 0\}$

② $$\iint_{\mathbb{R}^2} f_{XY}(x,y)\, dA = 1$$

<u>Expectation.</u> Suppose $g(X,Y)$

$$\mathbb{E}\, g(X,Y) = \iint_{\mathbb{R}^2} g(x,y)\, f_{XY}(x,y)\, dA.$$

↳ multivariable LOTUS.

<u>Joint CDF of X, Y</u>

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

$$= P\big((X,Y) \in (-\infty, x] \times (-\infty, y]\big)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v)\, dv\, du$$

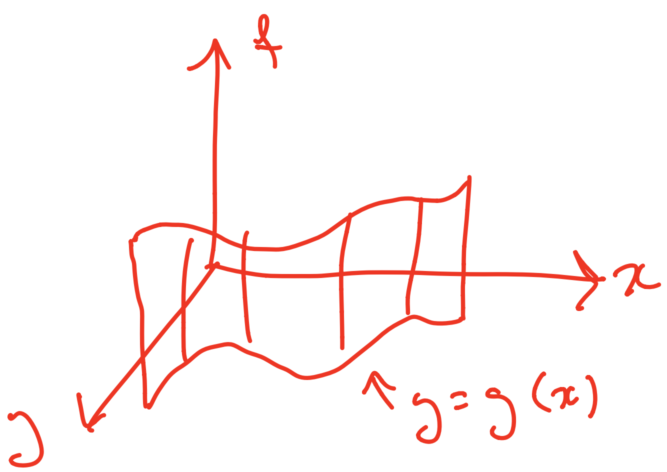
Relate the CDF to the PDF.

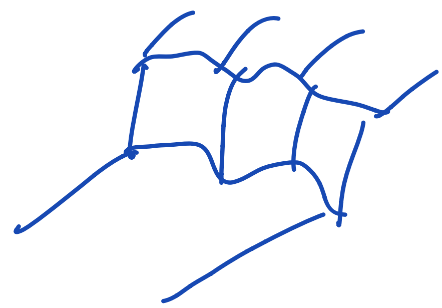
$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y).$$

Warning!! Two continuous RVs are not necessarily jointly continuous!

Ex X, $Y = g(X)$.

All of the probability is concentrated on the line $\{y = g(x)\} \subseteq \mathbb{R}^2$.





Ex X Y jointly continuous with PDF

$$f_{XY}(x,y) = \begin{cases} c(x+2y) & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$R_{(X,Y)} = [0,1] \times [0,1].$$

a) What is c?

b) What is the CDF?

c) What is $P(Y \le X)$?

a) $$1 = \iint_{\mathbb{R}^2} f_{XY}(x,y)\,dA = c \iint_{[0,1]\times[0,1]} (x+2y)\,dA.$$

$$= c \int_0^1 \int_0^1 (x+2y)\,dy\,dx$$

$$= c \int_0^1 \left(xy + y^2 \Big|_0^1 \right) dx$$

$$= c \int_0^1 x+1\,dx = c\left(\tfrac{1}{2}x^2 + x \Big|_0^1 \right.$$

$$= c\left(\tfrac{1}{2}+1\right) = c\,\frac{3}{2} = 1$$

$$c = 2/3.$$

b) choose $0 \le x \le 1$, $0 \le y \le 1$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v)\,dv\,du$$

$$= \frac{2}{3} \int_0^{x} \int_0^{y} (u + 2v)\, dv\, du$$

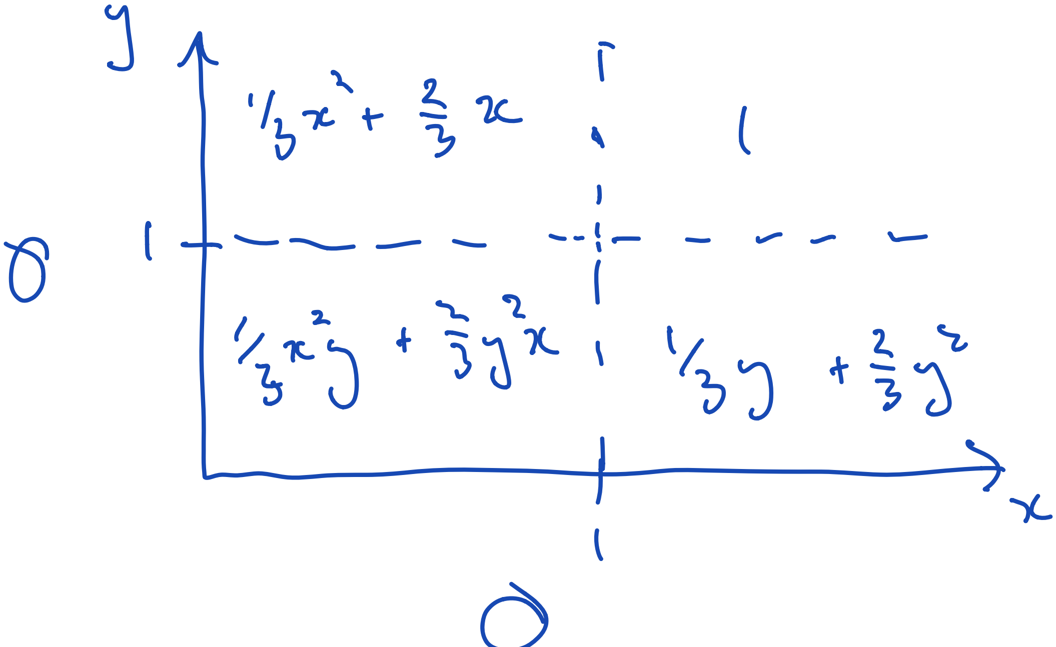
$$= \frac{2}{3} \int_0^x \left(uv + v^2 \Big|_0^y\right) du$$

$$= \frac{2}{3} \int_0^x uy + y^2 \, du$$

$$= \frac{2}{3} \left(\frac{1}{2} u^2 y + y^2 u \Big|_0^x \right)$$

$$= \frac{2}{3} \left(\frac{1}{2} x^2 y + y^2 x \right) = \frac{1}{3} x^2 y + \frac{2}{3} y^2 x.$$

$$F_{XY}(x,y) = \begin{cases} \frac{1}{3}x^2 y + \frac{2}{3} y^2 x & 0 \le x \le 1,\ 0 \le y \le 1 \\ \frac{1}{3} y + \frac{2}{3} y^2 & x \ge 1,\ 0 \le y \le 1 \\ \frac{1}{3} x^2 + \frac{2}{3} x & y \ge 1,\ 0 \le x \le 1 \\ 1 & x \ge 1,\ y \ge 1 \\ 0 & \text{otherwise.} \end{cases}$$



Marginals.

X, Y are jointly continuous RVs with $f_{XY}(x,y)$.

the X, Y marginals are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx.$$

Ex

$$f_{XY}(x,y) = \begin{cases} \frac{2}{3}(x+2y) & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ 0 & \end{cases}$$

$$f_X(x) = \int_0^1 \tfrac{2}{3}(x+2y)\,dy$$

$$= \tfrac{2}{3}\left(xy + y^2\Big|_0^1\right) = \frac{2}{3}(x+1)$$

$$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)\,dx = \frac{2}{3}\left(\tfrac{1}{2}x^2 + 2xy\Big|_0^1\right)$$

$$= \frac{2}{3}\left(\tfrac{1}{2} + 2y\right)$$

$$f_X(x) = \begin{cases} \frac{2}{3}(x+1) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(y) = \begin{cases} \frac{4}{3}y + \frac{1}{3} & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is $P(Y \le X)$?

$$P\big((X,Y) \in R\big) = \iint_R \tfrac{2}{3}(x+2y)\, dA.$$

$R = \{(x,y) : 0 \le y \le x \le 1\}$. Triangle.

$$= \int_0^1 \int_0^x \tfrac{2}{3}(x+2y)\, dy\, dx$$

$$= \tfrac{2}{3}\int_0^1 \left(xy + y^2 \Big|_0^x \right) dx = \tfrac{2}{3}\int_0^1 x^2 + x^2\, dx$$

$$= \tfrac{4}{9} x^3 \Big|_0^1 = 4/9.$$

Independence.

Recall: X, Y are independent.

if $P(X \leq x, Y \leq x) = P(X \leq x) P(Y \leq y)$.

or $F_{XY}(x,y) = F_X(x) F_Y(y)$.

Continuous? Take $\dfrac{\partial^2}{\partial x \partial y}$

$$\boxed{f_{XY}(x,y) = f_X(x) f_Y(y).}$$

independent.

Conditioning.

X, Y jointly continuous.

Conditional densities.

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}.$$

What does this give?

$$f_{XY} = \begin{cases} \frac{2}{3}(x+2y) & 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \\ 0 & \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{4}{3}y + \frac{1}{3} & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{2(x+2y)}{4y+1} & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$0 \le y \le 1$

This defines the following.

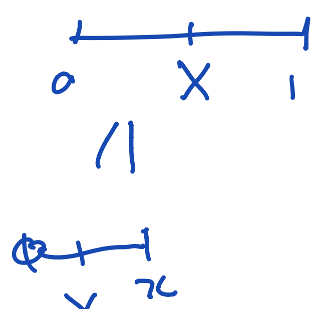
① $\mathbb{E}[X \mid Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y)\,dx$

② $\mathrm{Var}(X \mid Y=y) = \mathbb{E}[X^2 \mid Y=y] - \left(\mathbb{E}[X \mid Y=y]\right)^2.$

__Ex__

$X \sim \text{Uniform}(0,1).$

$Y \mid X=x \sim \text{Uniform}(0,x)$



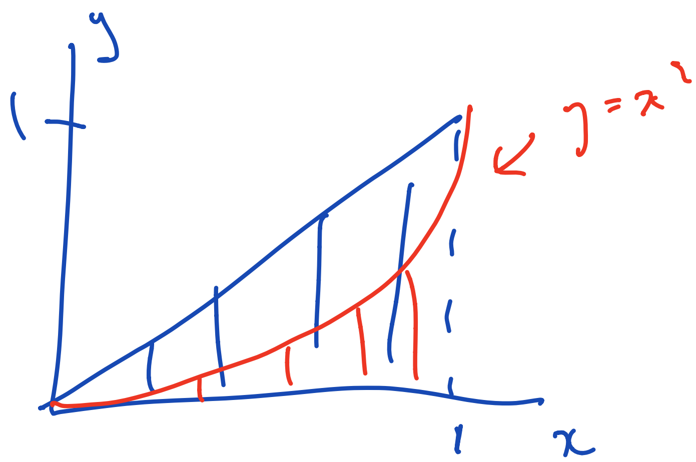
$P(Y \le X^2)$?

$$f_{Y|X}(y|x) = \begin{cases} 1/x & y \in [0, x] \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x)$$

$$= \begin{cases} 1/x & 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



$$\{ Y \le X^2 \}$$

$$P(Y \le X^2) = \int_0^1 \int_0^{x^2} 1/x \, dy \, dx$$

$$= \int_0^1 y/x \Big|_0^{x^2} dx$$

$$= \int_0^1 x \, dx = 1/2 .$$