Cauchy Schwarz

Let Y, Y be two RVs

then $$|\mathbb{E}XY| \le \sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2}$$

the equality $=$ holds if and only if

$X = aY$ for some $a \in \mathbb{R}$.

Proof

+ to get the other side.

$$0 \le \mathbb{E}\left(\frac{X}{\sqrt{\mathbb{E}X^2}} - \frac{Y}{\sqrt{\mathbb{E}Y^2}}\right)^2$$

$$= \mathbb{E}\left[\frac{X^2}{\mathbb{E}X^2} + \frac{Y^2}{\mathbb{E}Y^2} - \frac{2XY}{\sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2}}\right]$$

$$= \left(\underbrace{\frac{\mathbb{E}X^2}{\mathbb{E}X^2}}_{=1} + \underbrace{\frac{\mathbb{E}Y^2}{\mathbb{E}Y^2}}_{=1} - \frac{2\,\mathbb{E}XY}{\sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2}}\right) \Big/ 2$$

$$\Rightarrow \quad 1 - \frac{\mathbb{E}XY}{\sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2}} \ge 0.$$

CS $$\mathbb{E}XY \le \sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2}$$

$$+ \Rightarrow -\mathbb{E}XY \le \sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2}$$

If $\mathbb{E}XY = \sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2}$

then $\mathbb{E}\left(\frac{X}{\mathbb{E}X^2} - \frac{Y}{\mathbb{E}Y^2}\right)^2 = 0$

$$\Rightarrow \quad \underbrace{X - \left(\frac{\mathbb{E}X^2}{\mathbb{E}Y^2}\right)Y}_{=0} \quad\text{i.e.}\quad X = \underbrace{\left(\frac{\mathbb{E}X^2}{\mathbb{E}Y^2}\right)}_{= a > 0} Y$$

$\mathbb{E}XY = -\sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2}$

$$\Rightarrow \quad X = \underbrace{-\left(\frac{\mathbb{E}(X^2)}{\mathbb{E}(Y^2)}\right)}_{a < 0} Y$$

What does this mean for covariance?

$$\mathrm{Cov}(X, Y) = \mathbb{E}\,(\underbrace{X - \mu_X}_{U})(\underbrace{Y - \mu_Y}_{V})$$

Cauchy - Schwarz. $\Rightarrow$

$$|\mathrm{Cov}(X,Y)| \leq \sqrt{\underbrace{\mathbb{E}(X-\mu_X)^2}_{\mathrm{Var}(X)} \underbrace{\mathbb{E}(Y-\mu_Y)^2}_{\mathrm{Var}(Y).}}$$

$$\leq \sigma_X \sigma_Y$$

This is the same as.

$$|\rho(X,Y)| \leq 1$$

Sums of independent RVs

X, Y independent RVs.

$$Z = X + Y.$$

What is the distribution of Z?

Discrete

$$P_Z(z) = \sum_{x \in R_X} P_Y(z-x)\, P_X(x)$$

$$= P_Y * P_X(z)$$

discrete convolution.

Continuous case.

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(z-x)\, f_X(x)\, dx$$

$$= f_Y * f_X(z)$$

continuous convolution

"Proof"

Law of total probability.

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X = x)\, f_X(x)\, dx$$

$A = \{Z \leq z\}$

$$P(Z \leq z) = \int_{-\infty}^{\infty} P(\overset{\overset{Z}{\shortparallel}}{\underset{\underset{x}{\shortparallel}}{X} + Y} \leq z \mid X = x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} P(Y \leq z - x \mid X = x)\, f_X(x)\, dx$$

X, Y independent

$$= \int_{-\infty}^{\infty} F_Y(z-x)\, f_X(x)\, dx$$

take $\frac{d}{dz}$ $\Rightarrow$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(z-x)\, f_X(x)\, dx.$$

Ex $X_1 \sim \text{Poisson}(a_1)$
$X_2 \sim \text{Poisson}(a_2)$ independent.

$Z = X_1 + X_2$ $\quad z - x_1 \geq 0 \Downarrow x_1 \leq z.$

$$P_Z(z) = \sum_{x_1=0}^{z} \underbrace{\left(\frac{e^{-a_2} a_2^{z-x_1}}{(z-x_1)!} \right)}_{"X_2"} \underbrace{\left(\frac{e^{-a_1} a_1^{x_1}}{x_1!} \right)}_{"X_1"}$$

$$= \frac{e^{-a_1-a_2}}{z!} \sum_{x_1=0}^{z} \left(\frac{z!}{(z-x_1)!\,x_1!} \right) a_1^{x_1} a_2^{z-x_1}$$

$$= \frac{e^{-a_1-a_2}}{z!} \underbrace{\sum_{x_1=0}^{z} \binom{z}{x_1} a_1^{x_1} a_2^{z-x_1}}_{(a_1+a_1)^z}$$

$$= \frac{e^{-a_1-a_2}}{z!} (a_1+a_2)^z \sim \text{Poisson}(a_1+a_2)$$

"$(a+b)^n = \sum_{i=1}^{n} \binom{n}{i} a^i b^{n-i}$"

$\text{Poisson}(a_1) + \text{Poisson}(a_2) = \text{Poisson}(a_1 + a_2)$
$\uparrow$

independent.

Ex

$$X \sim N(\mu_X, \sigma_X^2)$$
$$Y \sim N(\mu_Y, \sigma_Y^2)$$ independent.

then $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

Probability Bounds.

Markov's inequality.

X is non-negative

$$P(X > a) \le \frac{EX}{a} \quad \text{for any } a > 0.$$

↳ used to show certain events unlikely!

Ex $X \sim \text{Binomial}(n, 1/2)$, $EX = n/2$

$$P(X > \tfrac{3}{4}n) \le \frac{EX}{\frac{3}{4}n} = \frac{n/2}{\frac{3}{4}n} = \frac{2}{3}$$

Chebyshev Inequality. X any RV.

$$P(|X-\mu_X| > b) \le \frac{\mathbb{E}|X-\mu_X|^2}{b^2} = \frac{\mathrm{Var}\,X}{b^2}$$

Ex $X \sim \mathrm{Binomial}(n, p)$

$$P\left(X > \frac{3n}{4}\right) \le P\left(|X - n/2| \le \frac{3n}{4} - \frac{n}{2}\right)$$

$$\le \frac{\mathrm{Var}(X)}{\left(\frac{3n}{2} - n/2\right)^2} = \frac{n(1/2)(1/2)}{n^2(1)}$$

$$= \frac{1}{4n}.$$