# Announcements

- Group Assignment 2
  - work out "Easy" Midterm Exam.
  - there is also a "Hard" Midterm Exam.

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## Recall

Consider

$$X_1, X_2, \dots, X_n \ - \ \text{i.i.d.}$$

$$\mu = \mathbb{E} X_i \quad , \quad \sigma^2 = \mathrm{Var}(X_i)$$

$$\bar{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

**WLLN**

$$\bar{X}_n \xrightarrow{P} \mu$$

in probability

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \varepsilon) = 0$$ for every $\varepsilon > 0$.

$$P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2 n} \to 0 \quad n \to \infty.$$

# CLT

Standardize.

$$\bar{Z}_n = \frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}} = \frac{X_1 + X_2 + \dots + X_n - n\mu}{\sqrt{n}\,\sigma}$$

$$\bar{Z}_n \xrightarrow{d} N(0,1) \quad \text{i.e.} \quad F_{\bar{Z}_n}(z) \to \Phi(z).$$

in distribution.

## How to use in practice.

$Y$, $P(a \le Y \le b)$

① $Y = X_1 + X_2 + \dots + X_n$, $\mu = \mathbb{E} X_i$

write as sum of i.i.d. $\sigma^2 = Var(X_i)$.

② $\mathbb{E} Y = n\mu$, $Var(Y) = n\sigma^2$

③ Standardize.

$$P(a \le Y \le b)$$

$$= P\left( \frac{a - n\mu}{\sqrt{n\sigma^2}} \le \underbrace{\frac{Y - n\mu}{\sqrt{n\sigma^2}}}_{= Z_n} \le \frac{b - n\mu}{\sqrt{n\sigma^2}} \right)$$

$n$ large.

$$\approx \Phi\left(\frac{b - n\mu}{\sqrt{n\sigma^2}}\right) - \Phi\left(\frac{a - n\mu}{\sqrt{n\sigma^2}}\right)$$

z - scores.

## Example

Send 1000 bit message.

There is noise, so each bit may be wrong with prob $p = 0.1$,

What is the prob that there are no more than 120 errors?

$$X_i \sim \text{Bernoulli}(0.1) = \begin{cases} 1 & \text{if bit } i \text{ has error.} \\ 0 & \text{otherwise.} \end{cases}$$

$$Y = X_1 + X_2 + \dots + X_{1000} = \#\text{ errors.}$$

$$\mathbb{E}Y = n\,\mathbb{E}X_i = \frac{n}{10}, \quad \text{Var}(Y) = n(.1)(.9) = \frac{9n}{100}$$

$$P(Y > 120) = P\left(\frac{Y - n/10}{\sqrt{9n/100}} > \frac{120 - n/10}{\sqrt{9n/100}}\right)$$

$n = 1000$

$$\frac{20}{3\sqrt{10}}$$

$$= 1 - \Phi\left(\frac{20}{3\sqrt{10}}\right)$$

$$\simeq 0.0175$$

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## Binomial Approx.

$$X_n \sim \text{Binomial}(n, p)$$

by the WLLN $\frac{X_n}{n} \xrightarrow{p} p \in [0,1]$

convergence in prob.

$$Z_n = \frac{X - np}{\sqrt{npq}} \xrightarrow{d} N(0,1).$$

Var(X).

How big to take $n$?

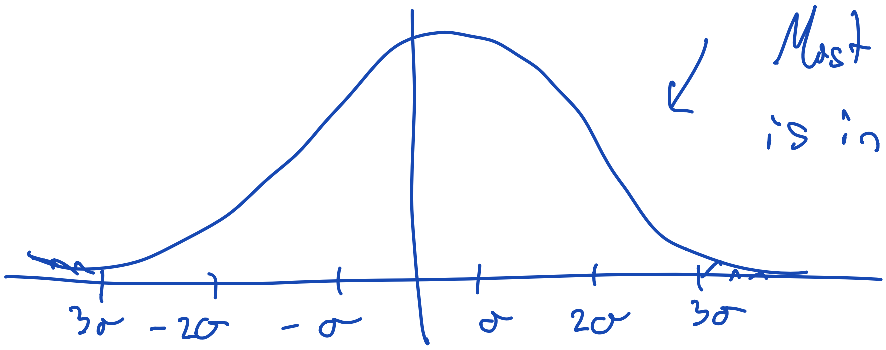
$$\mathbb{E}\frac{X_n}{n} \pm 3\sqrt{Var\left(\frac{X_n}{n}\right)} \in \{0,1\}.$$

$$\downarrow$$

$$p \pm 3\sqrt{\frac{pq}{n}} \in \{0,1\}.$$

by HW.

$$n \geq 6 \frac{\max\{p,q\}}{\min\{p,q\}}.$$




Most probability is in 3σ.

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## Convergence of RVs.

$$X_1, X_2, \ldots, X_n, \ldots$$

sequence of RVs.

How does $X_n$ approach $X$?

$$X_n \text{ "}\rightarrow\text{" } X \text{ ?}$$

## 3 types of convergence (that we will consider).

- Almost sure convergence.
- Convergence in probability.
- Convergence in distribution.

Almost sure. "with probability 1"

$$X_n \xrightarrow{a.s.} X \quad \text{as} \quad n \to \infty$$

if $$P\left( \lim_{n\to\infty} X_n = X \right) = 1 .$$

Convergence in Probability.

$$X_n \xrightarrow{P} X \quad \text{as} \quad n \to \infty. \quad \text{if}$$

$$\lim_{n\to\infty} P\left( |X_n - X| > \varepsilon \right) = 0 \quad \text{for all } \varepsilon > 0.$$

Convergence in Distribution.

$$X_n \xrightarrow{d} X \quad \text{as} \quad n \to \infty$$

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \text{for all } x \in \mathbb{R}.$$

Almost sure $\to$ Probability $\to$ Distribution.

## Properties

### ① Continuous mapping theorem

Let $g: \mathbb{R} \to \mathbb{R}$ be continuous.

and $X_n \to X$ almost sure or probability or distribution.

$$g(X_n) \to g(X).$$

"pass the limit inside $g$"

### ② Almost sure or Convergence in prob

if $X_n \to X$, $Y_n \to Y$ then

$$X_n Y_n \to XY.$$

products converge.

not true for convergence in distribution. !!

### ③ $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{d} c = \text{const.}$

then $X_n Y_n \xrightarrow{d} cX$

However, if one converges to a const.

then it does work.