Making Estimators Unbiased.

Recall: Given a random sample $X_1, X_2, \ldots, X_n$
an an estimator $\hat{\Theta}$ for Θ.

Bias: $B(\hat{\Theta}) = \mathbb{E}\hat{\Theta} - \Theta$

Can we make a biased estimator unbiased?

Suppose $B(\hat{\Theta}) \neq 0$ - biased.

Create new estimator $\hat{\Theta}_1 = c\,\hat{\Theta}$

$$
\begin{aligned}
B(\hat{\Theta}_1) &= \mathbb{E}\{c\hat{\Theta}\} - \Theta \\
&= c\,\mathbb{E}\hat{\Theta} - \Theta \\
&= c\{B(\hat{\Theta}) + \Theta\} - \Theta = 0
\end{aligned}
$$

what is c?

$$c = \frac{\Theta}{B(\hat{\Theta}) + \Theta}$$

$$\hat{\Theta}_1 = \frac{\Theta}{B(\hat{\Theta}) + \Theta}\,\hat{\Theta} \longrightarrow \text{unbiased.}$$

Ex Sample Variance.

$$X_1, X_2, X_3, \dots, X_n$$

$$\bar{S}_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - (\bar{X})^2$$

$$\mathbb{E}\bar{S}_n = \mathbb{E}X_i^2 - \mathbb{E}(\bar{X})^2$$

$$= Var(X_i) + \cancel{\mu^2} - Var(\bar{X}) - \cancel{\mu^2}$$

$$= Var(X_i) - Var(\bar{X})$$

$$= \sigma^2 - \frac{\sigma^2}{n}$$

$$\Rightarrow \quad B(\bar{S}_n) = -\frac{\sigma^2}{n}.$$

Biased.

Make unbiased.

$$c = \frac{\theta}{B(\bar{S}_n) + \theta} = \frac{\sigma^2}{-\frac{\sigma^2}{n} + \sigma^2} = \frac{n}{n-1}$$

Unbiased estimator for the variance.

$$\boxed{S^2 = \frac{n}{n-1} \bar{S}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$$

Maximum likelihood estimation.

Ex Container with 3 balls (red or white).

Goal: Estimate θ = # of red balls.

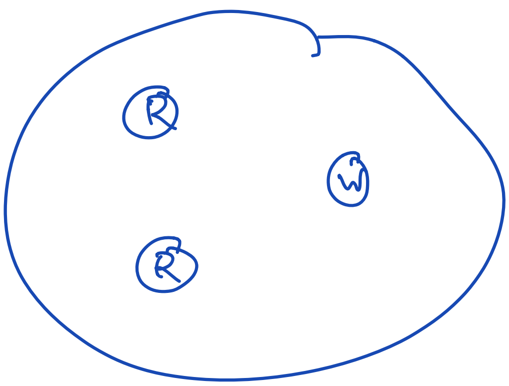
Sampling twice (w/o replacement).

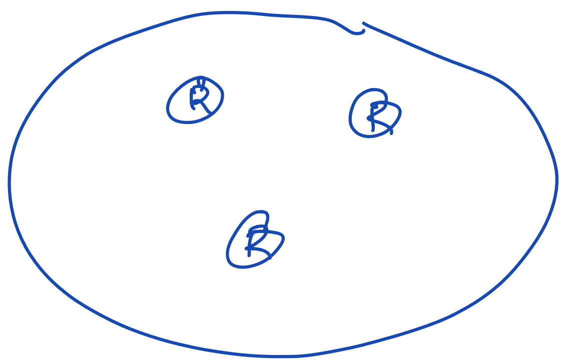
↳ get two reds.

What is the best estimate for θ?

Two options..

$\theta = 2$ $\theta = 3$





$P_{\theta=2}(2 \text{ reds}) = \frac{1}{\binom{3}{2}} = 1/3$ | $P_{\theta=3}(2 \text{ reds}) = 1$

$\Rightarrow \theta = 3$ is the most likely.

<u>Ex</u> Sampling is w/ replacement.

$\hookrightarrow$ Samples are independent.

Sample $n=4$ times. , X_1, X_2, X_3, X_4

$$X_i = \begin{cases} 1 & i^{th} \text{ sample red} \\ 0 & i^{th} \text{ sample white} \end{cases}$$

$X_i \sim \text{Bernoulli}(\theta/3)$

Suppose $X_1 = 1$, $X_2 = 0$, $X_3 = 1$, $X_4 = 1$,

$(1, 0, 1, 1)$

What is the most likely θ?

<u>Likelihood</u> $P(x_1, x_2, x_3, x_4 ; \theta)$

parameter.

$= P(x_1 ; \theta) P(x_2 ; \theta) P(x_3 ; \theta) P(x_4 ; \theta)$

↳ independence.

$$P(1,0,1,1;\theta) = \frac{\theta}{3}\left(1-\frac{\theta}{3}\right)\frac{\theta}{3}\frac{\theta}{3} = \left(\frac{\theta}{3}\right)^3\left(1-\theta/3\right)$$

θ	0	1	2	3
$P(1,0,1,1;\theta)$	0	0.0247	0.0988	0

↑ biggest.

$\theta = 2$ is the most likely.

General Method.

$$X_1, X_2, \ldots, X_n$$

random sample from a distribution with param. θ.

Def

$$L(x_1, x_2, \ldots, x_n; \theta) = \begin{cases} P_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n; \theta) & \leftarrow \text{discrete} \\ f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n; \theta) & \leftarrow \text{continuous} \end{cases}$$

Maximum Likelihood Estimator (MLE).

A maximum likelihood estimator

$$\hat{\theta}_{ML} = \hat{\theta}_{ML}(x_1, x_2, \dots, x_n)$$

is a value of θ that maximizes

$$\theta \mapsto L(x_1, x_2, \dots, x_n; \theta).$$

For a given random sample $X_1, X_2, \dots, X_n$

$$\hat{\Theta}_{ML} = \hat{\Theta}_{ML}(X_1, X_2, \dots, X_n).$$

<u>Remark</u>

$X_1, X_2, \dots, X_n$ independent.

$$L(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^{n} L(x_i; \theta)$$

$$L(x; \theta) = \begin{cases} p_X(x; \theta) \\ f_X(x; \theta). \end{cases}$$

Log-likelihood

$$\ln\left(L(x_1, x_2, \dots, x_n; \theta)\right) = \sum_{i=1}^{n} \ln\left(L(x_i; \theta)\right)$$

↳ easier to maximize.

Ex Exponential(θ)

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Likelihood.

$$L(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i}, \quad x_i \geq 0.$$

$$= \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

Take ln.

$$\ln\left(L(x_1, \ldots, x_n; \theta)\right) = n \ln \theta - \theta \sum_{i=1}^{n} x_i$$

Maximize ⇒ take derivative.

$$\frac{\partial}{\partial \theta} \ln\left(L(x_1, \ldots, x_n; \theta)\right) = \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0.$$

solve for θ.

$$\hat{\theta}_{ML} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i}$$ ↙ sample mean.

Works with multiple parameters

Ex $\quad X_1, X_2, \ldots, X_n \sim N(\theta_1, \theta_2)$

unknown.

Likelihood

$$L(x_1, x_2, \ldots, x_n; \theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_i-\theta_1)^2/2\theta_2} \right)$$

$$\ln\left(L(x_1, \ldots, x_n; \theta_1, \theta_2)\right) = \sum_{i=1}^{n} \left(-\ln\left(\sqrt{2\pi\theta_2}\right) - (x_i - \theta_1)^2 / 2\theta_2 \right)$$

$$= -\frac{n}{2} \ln(2\pi\theta_2) - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

Take derivatives.

$$\frac{\partial}{\partial\theta_1} : \quad \frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1) = 0$$

$$\frac{\partial}{\partial\theta_2} : \quad -\frac{n}{2\theta_2} + \frac{1}{2\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)^2 = 0.$$

System of eq in θ_1, θ_2

$$\Rightarrow \hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\theta}_1)^2$$

Biased.

Properties of MLEs.

Let $X_1, X_2, \ldots, X_n$ be a random sample for a distribution w/ parameter θ.

$$\hat{\Theta}_{ML} = \hat{\Theta}_{ML}(X_1, X_2, \ldots, X_n)$$

① $\hat{\Theta}_{ML}$ is consistent.

$$\hat{\Theta}_{ML} \xrightarrow[n \to \infty]{P} \theta$$

② Asymptotic unbias

$$\lim_{n \to \infty} B(\hat{\Theta}_{ML}) = 0,$$

③ CLT

$$z = \frac{\hat{\theta}_{ML} - \theta}{\sqrt{Var(\hat{\theta}_{ML})}} \xrightarrow{d} N(0, 1),$$