Announcements

- Reading Quiz 2 due Friday Canvas
- Homework 1 due Friday Gradescope.
  ↳ Don't start at the last minute!

Recall

Axioms of probability.

① $P(A) \geq 0$, $A \subset S$
② $P(S) = 1$
③ $A_1, A_2, \ldots$ countable collection $A_i \cap A_j = \phi$ $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Discrete (finite) Probability

Sample space $|S| < \infty$.

Ex Roll a die. (fair)

$$S = \{1, 2, 3, 4, 5, 6\}$$

Outcomes are equally likely

$$P(\{1\}) = P(\{2\}) = \dots = P(\{6\})$$

By normality and countable additivity.

$$1 = P(S) = P(\{1\} \cup \{2\} \cup \dots \cup \{6\})$$
$$= \sum_{i=1}^{6} P(\{i\}) = 6\,P(\{1\})$$

$$\Rightarrow \quad P(\{1\}) = 1/6 \quad \Rightarrow \quad p(i) \equiv P(\{i\}) = 1/6$$

$p(i)$: probability mass function.

Discrete probability for equally likely outcomes

$$P(A) = \frac{|A|}{|S|} \qquad (|A| - \#\text{ of outcomes})$$

↳ counting problem.

Composition Laws

① $P(A^c) = 1 - P(A)$, $A \subset S$

$\Rightarrow$ $S = A \cup A^c$ $1 = P(A \cup A^c) = P(A) + P(A^c)$

disjoint.

$\Rightarrow$ $P(\phi) = 0$, $\phi = S^c$

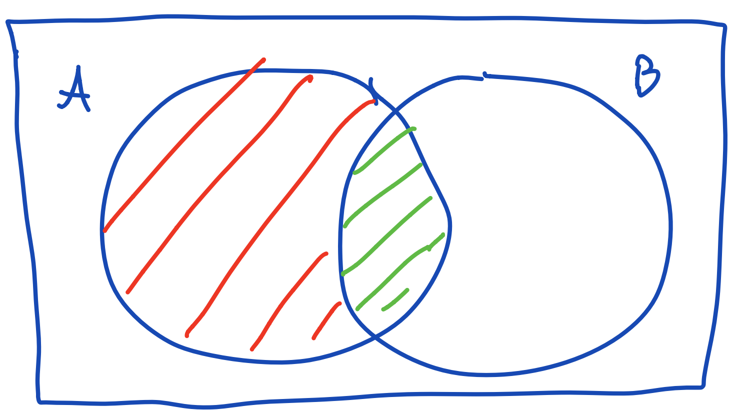
$\Rightarrow$ $0 \le P(A) \le 1$, $A \subset S$.

② Law of differences $A, B \subset S$

a. $P(A - B) = P(A) - P(A \cap B)$

b. $P(A - B) = P(A \cup B) - P(B)$

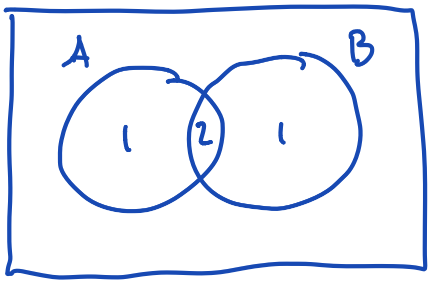
$\Rightarrow$ a. $A = (A - B) \cup (A \cap B)$ → disjoint.

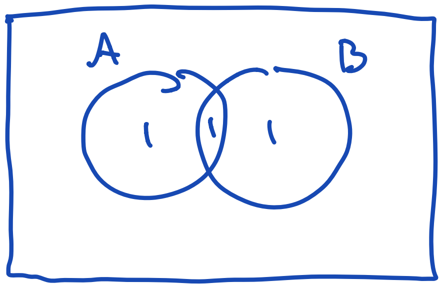


$P(A) = P(A - B) + P(A \cap B)$.

③ Law of addition (Inclusion-exclusion)

$$P(A\cup B) = P(A) + P(B) - P(A\cap B).$$

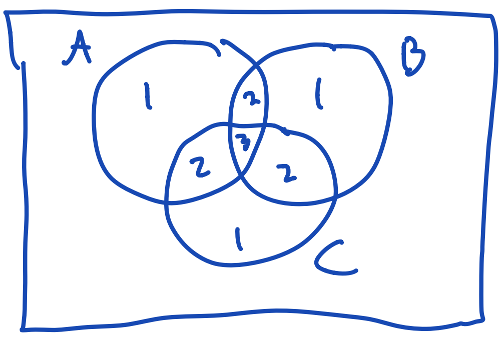


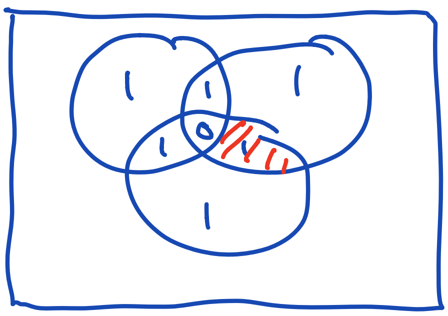


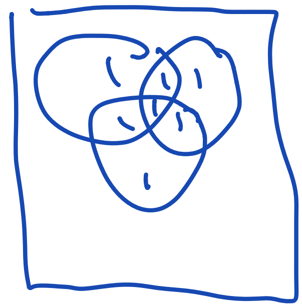
$P(A) + P(B) \quad - \quad P(A\cap B) \quad = \quad P(A\cup B)$

Three events?

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C)$$







$P(A) + P(B) + P(C) \quad - P(A\cap B) - P(B\cap C) - P(A\cap C) \quad + P(A\cap B\cap C)$

General formula

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} (-1)^{k+1}\left(\sum_{1\le i_1 < i_2 < \dots < i_k \le n} P(A_{i_1}\cap A_{i_2} \cdots \cap A_{i_k})\right)$$

Conditional Probability

- Probability is a model for lack of information.
- Sometimes new information comes to light, that can change the probabilities.

Ex

- $P(\text{rain}) = 0.3$
- $P(\text{rain} \mid \text{cloudy}) = 0.9$

↑ "given"

Ex Flip coin 3 times.

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Prob of 3 H $\qquad P(HHH) = \frac{1}{8}$

B = the first flip = H.

$P(HHH \mid B)$

$B = \{HHH, HHT, HTH, HTT\}$.

↳ treat this as new sample space.

$$P(HHH \mid B) = 1/4$$

More generally (finite events, equally likely)

$$P(A \mid B) = \frac{|A \cap B|}{|B|} = \frac{\frac{|A \cap B|}{|S|}}{|B|/|S|} = \frac{P(A \cap B)}{P(B)}$$

<u>Def</u> the probability of A given B.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \boxed{P(B) \neq 0.}$$

Conditional probability measures $P_B(A) \equiv P(A \mid B)$

are again probability measures. (axioms of probability).