Annoucements

- Final exam take home
- HW9 / HW10 combined due next Friday.
- Section 8.3 no longer covered.

Confidence intervals

Recall: $X_1, X_2, \dots, X_n$ with unknown mean $\theta = \mathbb{E}X_i$
known $\sigma^2 = \mathrm{Var}(X_i)$.

- If n is large.

$$Q = \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \sim N(0,1)$$

$\uparrow$ CLT.

$$P\left(\bar{X} - \frac{z_{\alpha/2}\sigma}{\sqrt{n}} \leq \theta \leq \bar{X} + \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) \geq 1-\alpha.$$

$$z_{\alpha/2} : \quad P(Z > z_{\alpha/2}) = \alpha/2$$

$$\Rightarrow \quad z_{\alpha/2} = \Phi^{-1}(1-\alpha/2).$$

What happens if you don't know the variance?

Two options:

① Find σ_{max}, s.t. $\sigma \le \sigma_{max}$.

② Use $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ as σ^2.

↳ estimating the variance.

Pivotn quantity.

$$Q = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

is this $\approx N(0,1)$?

now random.

Yes.:

$$Q = \underbrace{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}_{\xrightarrow{d} N(0,1)} \underbrace{\left(\frac{\sigma}{S}\right)}_{\xrightarrow{P} 1}$$

since S^2 is consistent estimator for σ^2.

$$\xrightarrow{d} N(0,1).$$

This means

$$\left[\bar{X} - \frac{z_{\alpha/2} S}{\sqrt{n}}, \bar{X} + \frac{z_{\alpha/2} S}{\sqrt{n}}\right]$$ is a $1-\alpha.\%$

confidence interval for μ (if n is large).

<u>Ex</u> (Opinion Polling) Estimate the fraction of people who plan to vote for a certain candidate. A

Random survey. $n = 100$, $X_1, X_2, \ldots, X_{n=100}$

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person plans to vote A.} \\ 0 & \text{otherwise.} \end{cases}$$

$X_i \sim \text{Bernoulli}(\theta)$

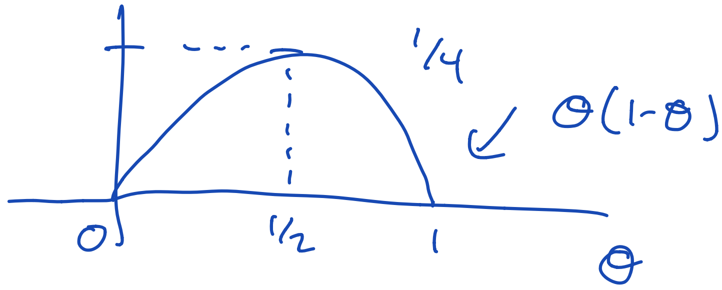
proportion of people who plan to vote for A.

What is a 95% confidence interval for θ?

$\bar{X}$ - estimator for θ. , $\mathbb{E}\bar{X} = \theta$

$\mathrm{Var}(\bar{X}) = \frac{\theta(1-\theta)}{n} \neq \sigma^2.$

However $\theta(1-\theta) \le 1/4$ (max $\theta = 1/2$).



Confidence interval is

$$\left[\bar{X} - \frac{z_{\alpha/2}\,\sigma_{max}}{\sqrt{n}},\ \bar{X} + \frac{z_{\alpha/2}\,\sigma_{max}}{\sqrt{n}} \right]$$

$$z_{\alpha/2} = z_{0.025} = 1.96$$

$$\left[\bar{X} - \frac{1.96}{2(10)},\ \bar{X} + \frac{1.96}{2(10)} \right]$$ is a 95%

confidence interval.

<u>Estimate the variance</u> (Normal Samples).

n, not large.

<u>Chi - squared distribution.</u>

Def $Z_1, Z_2, \dots, Z_n$ iid $N(0,1)$.

$$Y = Z_1^2 + Z_2^2 + \dots + Z_n^2 \sim \chi^2(n)$$

chi-squared distribution with n degrees of freedom.

$$\alpha, \quad \beta \qquad\qquad EY = \alpha/\beta$$

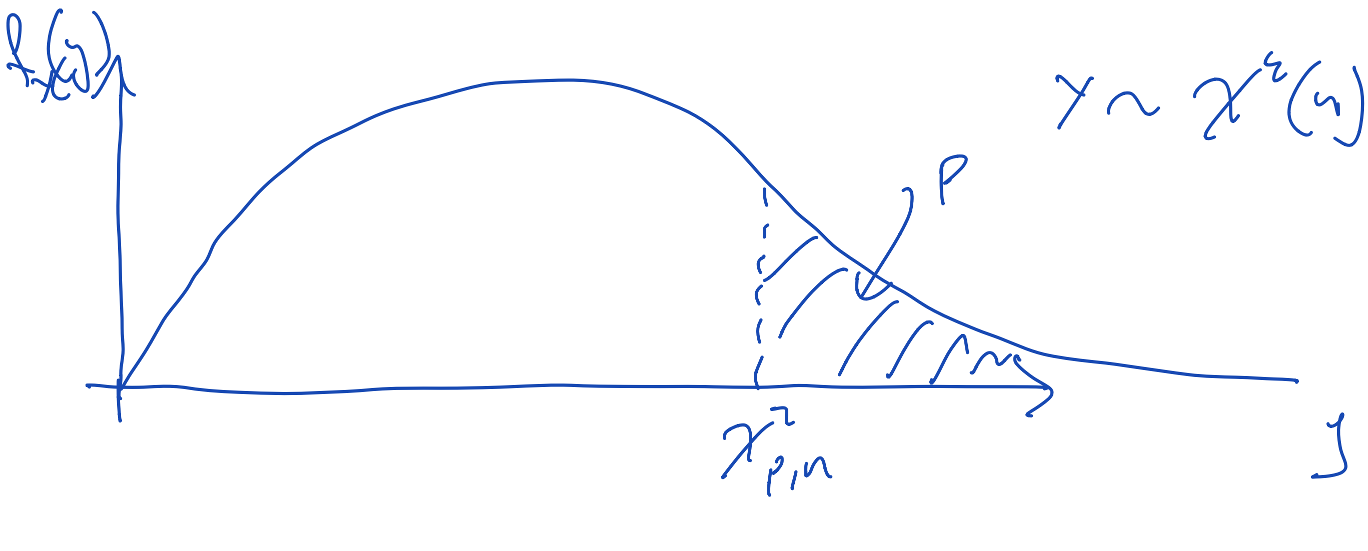
① $\chi^2(n) = \text{Gamma}\left(n/2, 1/2\right)$

$$f_Y(y) = \frac{1}{2^{n/2}\Gamma(n/2)} y^{n/2-1} e^{-y/2} \qquad y > 0$$

② $EY = n$, $\text{Var}(Y) = 2n$.

③ $\chi^2_{p,n}$ the number such that

$$P\left(Y > \chi^2_{p,n}\right) = p$$



Why is this useful?

Theorem Let $X_1, X_2, \dots, X_n$ iid $N(\mu, \sigma^2)$

then $Y = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{(n-1)S^2}{\sigma^2}$

$$\sim \chi^2(n-1)$$

one less d.o.f.

- Moreover S^2 and $\bar{X}$ are <u>independent</u>

Estimate intervals for the variance

Suppose $X_1, X_2, \dots, X_n$ iid $N(\mu, \sigma^2)$.

- Find $1-\alpha$ % confidence interval for σ^2.

① Pivotal quantity.

$$Q = \frac{(n-1)}{\sigma^2} S^2 = 1/\sigma^2 \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2(n-1)$$

② $1-\alpha$ interval for Q.

$$P\left(\chi^2_{1-\alpha/2,\,n-1} \le Q \le \chi^2_{\alpha/2,\,n-1}\right) = 1-\alpha.$$

③ $$\Rightarrow \chi^2_{1-\alpha/2,\,n-1} \le \frac{n-1\ S^2}{\sigma^2} \le \chi^2_{\alpha/2,\,n-1}$$

$$\Rightarrow \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,\,n-1}}, \quad \sigma^2 \ge \frac{(n-1)S^2}{\chi^2_{\alpha/2,\,n-1}}$$

$$\Rightarrow \frac{(n-1)S^2}{\chi^2_{\alpha/2,\,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,\,n-1}}$$

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,\,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,\,n-1}}\right]$$ $1-\alpha$ % confidence interval for σ^2.

How to look up $\chi^2_{\alpha/2,\,n-1}$.

Ex

$n=16$, $\alpha = 0.05$, $\alpha/2 = 0.025$.

$\chi^2_{0.025,\,9.} = 19.023$, $\chi^2_{1-0.025,\,9} = 2.700.$

$= .975, 9$