Last Lecture! → Next week review.

Recall: χ^2-distribution.

$$X_1, X_2, \dots, X_n \text{ iid } N(0,1)$$

HW9

$$Y = X_1^2 + X_2^2 + \dots + X_n^2 \sim \chi^2(n) = \text{Gamma}(n/2, 1/2)$$

"Chi-squared w/ n degrees of freedom"

Useful because $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$

$$Q = \frac{(n-1)}{\sigma^2} S^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2(n-1)$$

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t-distribution. (Students t-distribution).

↳ Discovered by William Gosset 1908
- under the pseudonym "student".

Def Let $Z \sim N(0,1)$, $Y \sim \chi^2(n)$

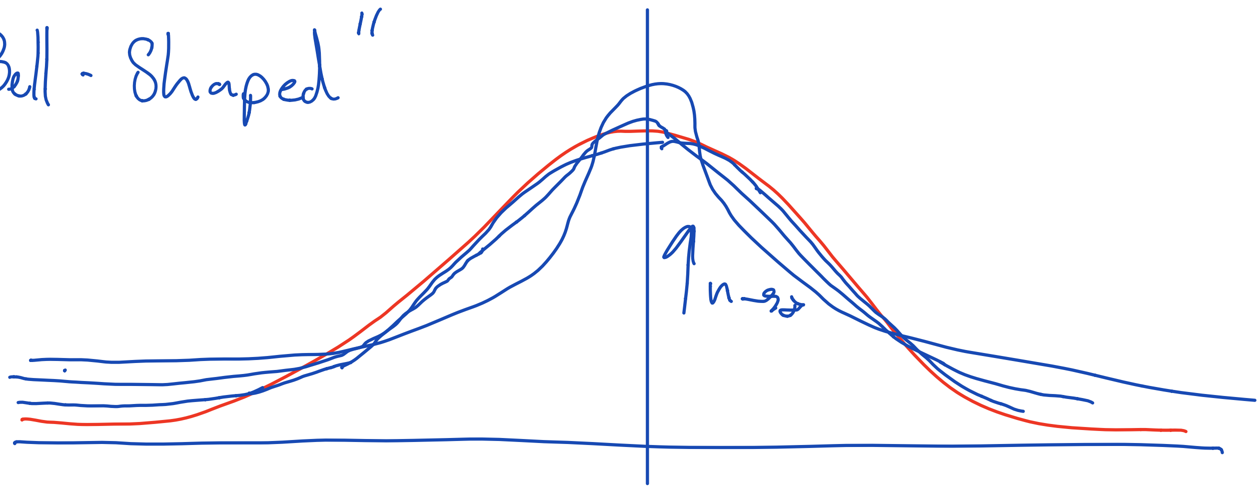
then $T = \dfrac{Z}{\sqrt{Y/n}} \sim T(n)$

is a random variable satisfy a t-distribution with n-degree's of freedom.

Properties: ① PDF

$$f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} \quad -\infty < t < \infty.$$

"Bell - Shaped"



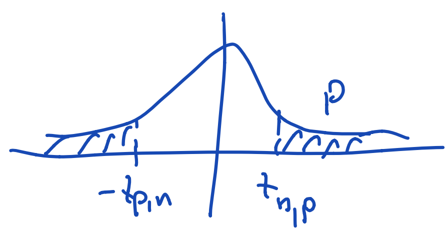
② $\mathbb{E}T = 0$, $Var(T) = \frac{n}{n-2}$

$n > 1$ $n > 2$

③ CLT $T(n) \xrightarrow{d} N(0,1)$ $n \to \infty$

④ Let $t_{p,n}$ be such that.

$$P(T > t_{p,n}) = p$$



Symmetry $t_{1-p,n} = -t_{p,n}$

MATLAB

$$t_{p,n} = F^{-1}_{T(n)}(1-p) = \text{tinv}(1-p, n)$$

Also - table on course webpage.

<u>Main Application.</u>

$$X_1, X_2, X_3, \dots, X_n \sim N(\mu, \sigma^2)$$

Find confidence interval for μ.

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim T(n-1)$$

note one-less.

<u>Proof</u>

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$Y = \frac{n-1}{\sigma^2} S^2 \sim \chi^2(n-1)$$

so,

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{Z}{\sqrt{Y/n-1}} \sim T(n-1).$$

Find $(1-\alpha)\%$ confidence interval for μ.

① $Q = \dfrac{\bar{X} - \mu}{S/\sqrt{n}} \sim T(n-1)$: Pivot.

② $P\left(-t_{\alpha/2, n-1} \le Q \le t_{\alpha/2, n-1}\right) = 1-\alpha$

$\quad \hookrightarrow t_{\alpha/2, n-1} = F^{-1}_{T(n-1)}\left(1-\alpha/2\right)$

③ Algebra

$$P\left(\bar{X} - \frac{t_{\alpha/2,n-1} S}{\sqrt{n}} \le \mu \le \bar{X} + \frac{t_{\alpha/2,n-1} S}{\sqrt{n}}\right) = 1-\alpha$$

$\left[\bar{X} - \frac{t_{\alpha/2,n-1} S}{\sqrt{n}}, \bar{X} + \frac{t_{\alpha/2,n-1} S}{\sqrt{n}}\right]$, $1-\alpha$% confidence interval.

Ex: A farmer weigh 10 watermelons.

7.72, 9.58, 12.38, 2.77, 11.27
8.10, 11.10, 7.80, 10.17, 6.00

Assume weights are normally distributed.

95% confidence interval on μ, σ^2?

① Numerically calculate.

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = 9.26 \quad , \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = 3.96.$$

② $n = 10$, $\alpha = 1 - .95 = 0.05$, $\alpha/2 = 0.025$.

$t_{\alpha/2, n-1} = 2.262$

$\chi^2_{\alpha/2, n-1} = 19.023$

$\chi^2_{1-\alpha/2, n-1} = 2.700$

Mean: μ $\left[\bar{X} - \frac{t_{\alpha/2,n-1} S}{\sqrt{n}} , \bar{X} + \frac{t_{\alpha/2,n-1} S}{\sqrt{n}}\right]$

$$= \left[9.26 - \frac{2.266\sqrt{3.96}}{\sqrt{10}} , 9.26 + \frac{2.266\sqrt{3.96}}{\sqrt{10}}\right]$$

$$= [7.834 , 10.686] \quad 95\% \text{ confidence}$$

Variance σ^2: $\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} , \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right]$

$$= \left[\frac{9(3.96)}{19.023} , \frac{9(3.96)}{2.700}\right]$$

$$= [1.874 , 13.2]$$

What about other distributions?

↳ Can be challenging. → Here is an example

<u>Ex</u> $X_1, X_2, \ldots, X_n \sim \text{Exponential}(\theta)$

What is a $(1-\alpha)\%$ confidence interval for θ?

Recall $\mathbb{E} X_i = 1/\theta$ $\quad f_X(x) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

What is an estimator for θ?

<u>MLE</u>

$$L(x_1, x_2, \ldots, x_n ; \theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$\ln(L(x_1, x_2, \ldots, x_n ; \theta)) = n \ln \theta - \theta \sum_{i=1}^{n} x_i$$

$$\frac{\partial}{\partial \theta} \rightarrow \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{\frac{1}{n} \sum_{i=1}^{n} X_i} = 1/\bar{X}.$$

Confidence interval?

Pivot, $X_1 + X_2 + \dots + X_n \sim \text{Gamma}(n, \theta)$

$$\frac{X_i}{\theta} \sim \text{Exponential}(1)$$

$$Q = \frac{X_1 + X_2 + \dots + X_n}{\theta} = \frac{n\bar{X}}{\theta} \sim \text{Gamma}(n, 1)$$

Define. $\gamma_{p,n}$ $\quad P(Q > \gamma_{p,n}) = p$

$$\gamma_{p,n} = F^{-1}_{\Gamma(n,1)}(1-p) = \text{gaminv}(1-p, n)$$

↑ MATLAB.

$$P(\gamma_{1-\alpha/2,n} \leq Q \leq \gamma_{\alpha/2,n}) = 1-\alpha$$

Algebra.

$$\gamma_{1-\alpha/2,n} \leq \frac{n\bar{X}}{\theta} \leq \gamma_{\alpha/2,n}.$$

$$\theta \leq \frac{n\bar{X}}{\gamma_{1-\alpha/2,n}}, \qquad \theta \geq \frac{n\bar{X}}{\gamma_{\alpha/2,n}}$$

$$P\left(\frac{n\bar{X}}{\gamma_{\alpha/2,\,n}} \leq \theta \leq \frac{n\bar{X}}{\gamma_{1-\alpha/2,\,n}}\right) = 1-\alpha.$$

$(1-\alpha)\%$ confidence interval for θ.