Lecture 5

Independence and Bayes Rule

Recall: A, B are independent if

$$P(A \cap B) = P(A)\,P(B).$$

- Knowledge about one event doesn't affect the probability of the other.

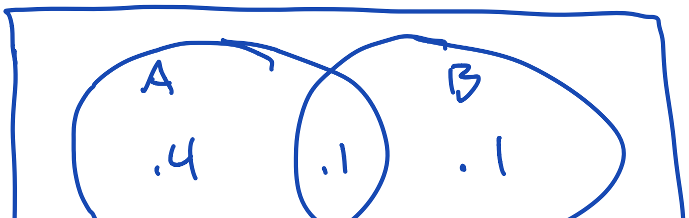
Equivalent to

- $P(A|B) = P(A)$ if $P(B) \neq 0$
- $P(B|A) = P(B)$ if $P(A) \neq 0$.

Mutually exclusive events (not probability 0) cannot be independent.

- $P(A \cap B) = 0 \neq P(A)P(B)$

Ex



$P(A \cap B) = .1$, $P(A) = .4 + .1 = .5$

$P(B) = .1 + .1 = .2$

$P(A)P(B) = (.5)(.2) = .1 = P(A \cap B)$ ✓

They are independent.

Mutiple sets?

A, B, C are independent if.

- $P(A \cap B) = P(A)P(B)$
- $P(B \cap C) = P(B)P(C)$
- $P(C \cap A) = P(C)P(A)$
- $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Properties

If A, B are independent. , then.

- A, B^c independent
- A^c, B independent
- A^c, B^c independent.

- $P(\underset{A-B}{A\cap B^c}) = P(A-B) = P(A) - P(A\cap B)$

$$= P(A) - P(A)P(B)$$

$$= P(A)(1 - P(B))$$
$$= P(A)P(B^c).$$

- $P(A^c\cap B^c) = P(A^c - B) = P(A^c) - P(A^c\cap B)$

$$= P(A^c) - P(A^c)P(B)$$
$$= P(A^c)P(B^c).$$

Bayes Theorem (Rule).

Recall $P(A\cap B) = P(A|B)P(B)$

$$= P(B|A)P(A).$$

Divide by $P(A)$ $(P(A)\neq 0)$. then

$$\boxed{P(B|A) = \frac{P(A|B)P(B)}{P(A)}}$$

Bayes Rule.

Bayesian inference

$$P(B\mid A) = \left[\frac{P(A\mid B)}{P(A)}\right] P(B).$$

↑ support that A gives B.

↑ prior

General version.

$B_1, B_2, \dots, B_n$ partition,

$$P(B_i\mid A) = \frac{P(A\mid B_i)\,P(B_i)}{\sum_{k=1}^{n} P(A\mid B_k)\,P(B_k)}.$$

Example

3 bags

1: 75 red, 25 blue

2: 60 red, 40 blue

3: 45 red, 55 blue

Choose a bag a random and pick a ball.

- What is the probability the ball is red.?
- Given you picked red, what is the probability you picked bag 1?

B_i - event bag i is picked.

R - event you pick a red ball

$$P(B_1 \mid R) = \frac{P(R \mid B_1)P(B_1)}{P(R \mid B_1)P(B_1) + P(R \mid B_2)P(B_2) + P(R \mid B_3)P(B_3)}$$

$$= \frac{\left(\frac{75}{100}\right)\cancel{\left(\frac{1}{3}\right)}}{\left(\frac{75}{100}\right)\cancel{\left(\frac{1}{3}\right)} + \left(\frac{60}{100}\right)\cancel{\left(\frac{1}{3}\right)} + \left(\frac{45}{100}\right)\cancel{\left(\frac{1}{3}\right)}}$$

$$= \frac{.75}{.75 + .6 + .45} = \frac{5}{12} > 1/3$$

Base rate rate fallacy.

Testing a rare disease.

- Disease affects 1/10,000
- False positive rate = 2%
- False negative rate = 1%

What is the you have the disease, given that you test positive?

- D - you have the disease
- T - you test positive

$$P(D \mid T) = \frac{P(T \mid D)\, P(D)}{P(T \mid D)\, P(D) + P(T \mid D^c)\, P(D^c)}$$

$$= \frac{\left(1 - P(\overset{''\ .01}{T^c \mid D})\right) \frac{1}{10,000}}{(1 - .01)\frac{1}{10,000} + .02\left(1 - \frac{1}{10,000}\right)}$$

$$= \frac{99}{99 + \boxed{2(9,999)}}$$

$99 + \boxed{2(9,999)}$ → contribution from all the people with false positives.