Announcements

- Sign up for the right conference sections in CAB by Friday.
  - see campuswire post.

Recall: Counting.

- sampling — Draw k things out of n.
  - Ordered — assign trophies
    - with replacement
    - without replacement
  - Unordered — draw a poker hand.
    - with replacement
    - without replacement.

Ordered w/o replacement. $P^n_k = \frac{n!}{(n-k)!}$

Birthday Problem

20 people, What is the probability that at least two share the same birthday?

$A =$ at least two share the same birthday.

$A^c =$ no two people share the same birthday.

Assign birthdays to people

— 20

365   364   363   . . . $365-19$

$= 346$

$$|A^c| = 365 \times 364 \times 363 \times \cdots \times 346.$$

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|S|}$$

$$|S| = 365^{20}$$

$$P(A) = 1 - \frac{365 \times 364 \times \cdots \times 346}{365^{20}}.$$

$$\approx .41$$

Unordered Sampling without replacement.

$A = \{1, 2, 3\}$, $k = 2$, $n = 3$.

$\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ - 3 elements.

In general.

Recall: ordered: $P^n_k = \dfrac{n!}{(n-k)!}$

to get unordered, divide by all the ways to permute the k things you draw.

Binomial coefficient.

$$C^n_k = \binom{n}{k} = \frac{P^n_k}{k!} = \frac{n!}{k!\,(n-k)!}$$

↑ "n choose k"

the number of ways to choose k elements out of n elements (unordered).

C^3_2	P^3_2	
$\{1,2\}$	$(1,2)$	$(2,1)$
$\{1,3\}$	$(1,3)$	$(3,1)$
$\{2,3\}$	$(2,3)$	$(3,2)$

$$P^3_2 = 2!\, C^3_2$$

Ex Draw 5 cards out of 52 card deck.
How many hands?

$$\binom{52}{5} = \frac{52!}{5!(47)!}$$

$$= \frac{52\times 53\times \cdots 48}{5\times 4\times 3\times 2\times 1}\times\left(\frac{47\times\cdots}{47!}\right)$$

Symmetry of # of combinations.

$$\binom{n}{k} = \binom{n}{n-k}$$

"Proof"

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!}$$

$$= \binom{n}{n-k}.$$

Pascal's Rule

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascals Triangle

$$\begin{array}{ccccccccccc}
&&&&1&&&& & - & n=0\\
&&&1&&1&&& & - & n=1\\
&&1&&2&&1&& & - & n=2\\
&1&&3&&3&&1& & - & n=3\\
1&&4&&6&&4&&1 & - & n=4\\
\uparrow&&\uparrow&&\uparrow&&\uparrow&&\uparrow&&\\
k=0&&k=1&&k=2&&k=3&&k=4&&\\
\uparrow&&\uparrow&&\uparrow&&\uparrow&&\uparrow&&\\
\binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4}&&
\end{array}$$

Binomial Coefficient??

Lemma

$$(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$$

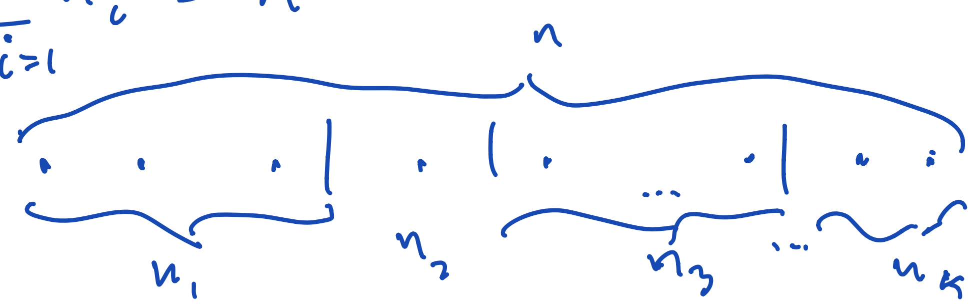
Binomial expansion.

Multinomial Coefficient.

Number of partitions

What is the number of ways to partition a set of n elements into k- disjoint subsets of size. $n_1, n_2, n_3, \ldots n_k$.

$$\sum_{i=1}^{k} n_i = n$$



$$\binom{n}{n_1 \; n_2 \; n_3 \ldots n_k} = \frac{n!}{n_1! \; n_2! \; n_3! \ldots n_k!}$$

space

Example How many unique rearrangements of "RACECAR" is there?

$$\binom{\overset{\swarrow \text{ letters}}{7}}{2 \;\; 2 \;\; 2 \;\; 1} = \frac{7!}{2!(2!)(2!)(1!)}.$$

Sample unordered w/ replacement.

$A = \{1, 2, 3\}, \quad k = 2, \quad n = 3$

$\{1,1\}, \{2,2\}$
$\{1,2\}, \{2,3\}$ — 6 ways.
$\{1,3\}, \{3,3\}$

$$\underbrace{\cdots}_{n_1}\ \underbrace{\cdots}_{n_2}\ \underbrace{\cdots}_{n_3} = K$$

General formula

Number of ways to sample K things out of n, unordered with replacement.

Correction.

$$\left[\binom{n+K-1}{K} = \frac{(n+K-1)!}{K!\,(n-1)!}\right]$$

Idea: Let x_1, x_2, x_3 = # 1, 2, 3's respectively.

Note $x_1 + x_2 + x_3 = 2$,

Ex:

Ex:

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ \{1,1\} \longrightarrow & (2, & 0, & 0) & \longrightarrow 2+0+0=2 \\ \{1,2\} \longrightarrow & (1, & 1, & 0) & \longrightarrow 1+1+0=2 \\ \{1,3\} \longrightarrow & (1, & 0, & 1). & \longrightarrow 1+0+1=2. \\ \{2,3\} \longrightarrow & (0, & 1, & 1) & \longrightarrow 0+1+1=2 \\ \{2,2\} \longrightarrow & (0, & 2, & 0) & \quad 0+2+0=2 \\ \{3,3\} \longrightarrow & (0, & 0, & 2) & \quad 0+0+2=2 \end{array}$$

The number of ways to sample k things out of n, unordered with replacement is the same as the number of solutions to the equation.

$$x_1 + x_2 + \dots + x_n = k \quad , \; x_i \in \{0,1,\dots k\}.$$

This is given by $\binom{n+k-1}{k}$.

Proof

Consider the rule (numbers to vertical lines)

$$\begin{aligned} 1 &\longrightarrow | \\ 2 &\longrightarrow || \\ &\vdots \\ n &\longrightarrow \underbrace{||\dots|}_{n} \end{aligned}$$

Then e.g. $\quad 0 + 2 + 1 \longrightarrow +||+|$

Then every "solution" is just a reshuffling of

k - bars "|" and $n-1$ - plus signs "+".

There are $\binom{k+n-1}{k}$ total ways to do this.