Discrete Random Variables

What is a random variable (R.V.)?

↳ Often intersested in assigning a number ($\mathbb{R}$) to an outcome.

Def A random variable (R.V.) X is rule for assigning a real number to an outcome of an experiment.

Math lingo: $X : S \longrightarrow \mathbb{R}$. - function.

Def: Range of X, R_X, Range(X) is all the possible values that X can take.

Ex: Toss a coin 4 times. How many heads?

$X = \#$ heads

$S = \{TTTT, TTTH, \ldots, HHHH\}$.

$X(\{TTTT\}) = 0$
$X(\{TTTH\}) = 1$
$\vdots$

$\Rightarrow \quad R_X = \{0, 1, 2, 3, 4\}$

Ex T - lifetime of a certain product.

$$R_T = \mathbb{R}_+ = [0, \infty).$$

Def X is a discrete R.V. if it's range is countable.

$$R_X = \{x_1, x_2, x_3, \dots\}. \quad x_i \in \mathbb{R}.$$

Define events through random variables

$$A \subset \mathbb{R}.$$

Def: $\{X \in A\} = \{s \in S : X(s) \in A\}.$

Ex Coin toss (4 times)

$$\{X \in \{0, 3\}\} = \{TTTT, HHHT, HHTH, HTHH, THHH\}.$$

$$P(X \in A) = P(\{s \in S : X(s) \in A\})$$

Probability mass function.

Def X is a discrete R.V.

$$R_X = \{x_1, x_2, \dots\}.$$

the function $P_X : R_X \to [0, 1]$.

$$\boxed{P_X(x_k) = P(X = x_k).}$$

$$\begin{tikzcd} X : S \arrow[r] \arrow[d, leftrightarrow] & R_X \arrow[d, leftrightarrow] \\ P \arrow[r] & P_X \end{tikzcd}$$

Remark: $R_X = \{x_1, x_2, \dots\}$.

Let $\{X = x_1\}, \{X = x_2\}, \dots$
form a partition of S.

$$S = \bigcup_{i=1}^{\infty} \{X = x_i\}$$

disjoint.

$$1 = P(S) = \sum_{i=1}^{\infty} P(X = x_i) = \sum_{i=1}^{\infty} P_X(x_i).$$

$A \subseteq \mathbb{R}.$

$$P_X(A) = \sum_{\substack{x \in A \\ x \in R_X}} P_X(x_i).$$

Properties of $P_X(x)$.

a) $0 \leq P_X(x) \leq 1 \quad , x \in R_X$

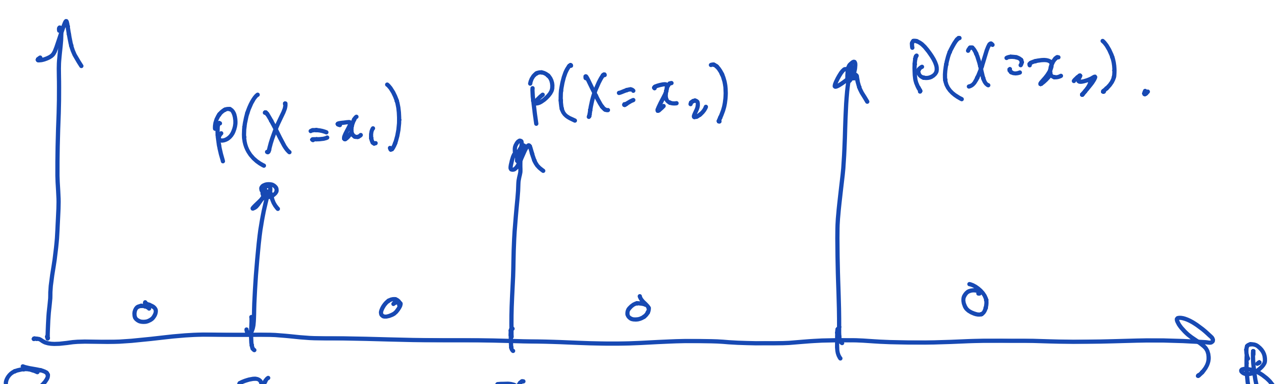
b) $\displaystyle\sum_{x_k \in R_X} P_X(x_k) = 1$

c) $A \subseteq R_X$

$$P_X(A) = P(X \in A) = \sum_{x \in A} P_X(x).$$

Remark $P_X(x)$ can always be defined for $x \in \mathbb{R}$

$$P_X(x) = P(\{X = x\}) = \begin{cases} P_X(x) \ , & x \in R_X \\ 0 & \text{otherwise.} \end{cases}$$



$x_1 \quad x_2 \quad x_3 \quad \ldots$

Ex Toss a fair coin twice.

$X = \#$ of heads.

What is R_X, $P_X(x)$?

$S = \{TT, TH, HT, HH\}$, $R_X = \{0, 1, 2\}$.

x	0	1	2
$P_X(x)$	1/4	1/2	1/4

$= 1/4 + 1/2 + 1/4 = 1$

$$P_X(k) = \binom{2}{k} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{2-k} = \frac{\binom{2}{k}}{4}$$

↳ Binomial Distribution.

Ex Roll a die over and over until I get a 6.

$X = \#$ of roll until you get a 6.

↳ includes the last roll

$R_X = \{1, 2, 3, 4, \ldots\} = \mathbb{N}$.

$$P_X(k) = P(X=k) = \left(\frac{5}{6}\right)^{k-1} \frac{1}{6}$$

probability of rolling not 6 k-1 times

roll a 6.

Distribution : Geometric distribution.

Independence of RVs.

<u>Def</u>: Consider two discrete R.V.s X, Y.
Then X, Y are independent if.

$\{X=x\}, \{Y=y\}$ are independent for all $x, y \in \mathbb{R}$.

$$P(X=x, Y=y) = P(X=x)\,P(Y=y).$$

joint distribution.

<u>More generally</u>: n discrete RVs.

$X_1, X_2, \dots, X_n$ are independent if.

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

$$= P(X_1 = x_1)\, P(X_2 = x_2) \ldots P(X_n = x_n).$$

for all $x_1, x_2, \ldots, x_n$