Announcements

- Group assignment this week. Meet with your 3-4 group members during your conference section time and complete the discussion / exercise.
  - ↳ Will release group assignment in Gradescope on Wednesday.
  - ↳ Graded for completion.
  - ↳ May get to design an exam problem!

Special Distributions

- Special families of random variables.
- The pmf is called "distribution".

Bernoulli R.V. - A random variable made from a Bernoulli trial.

- **A Bernoulli Trial**: A experiment with two outcomes. $\{S, F\}$ - success and failure.

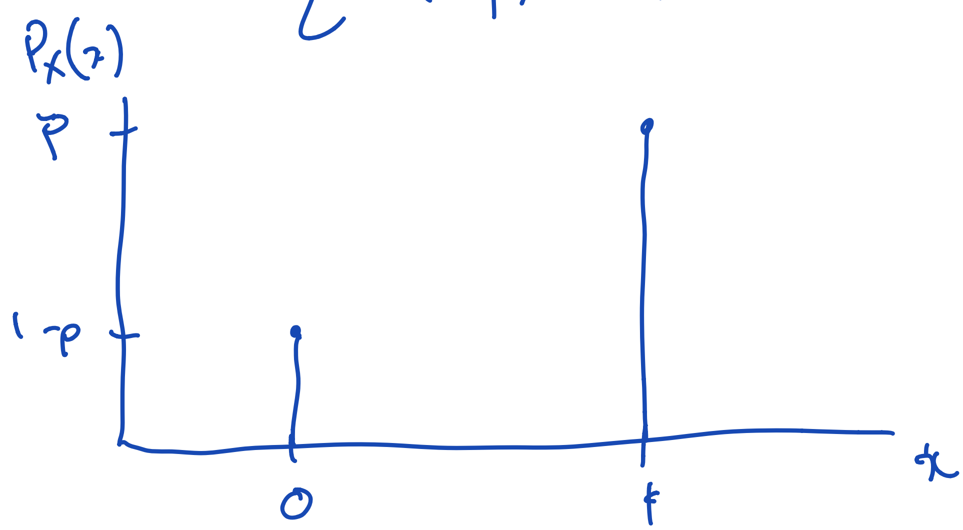
but can be more general $\{H, T\}$, $\{\text{in } A, \text{not in } A\}$.

<u>Def</u> $X \sim \text{Bernoulli}(p)$ if $R_X = \{0, 1\}$,

↳ distributed as.

or has pmf

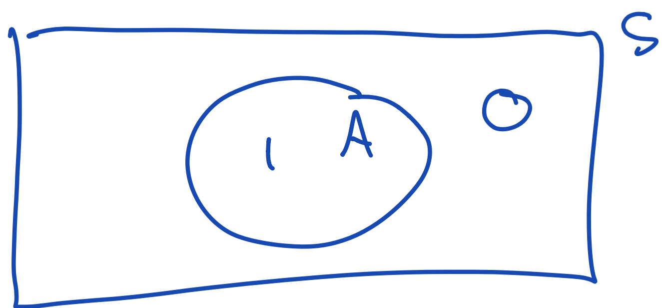
$$P_X(k) = \begin{cases} p & k = 1 \\ (1-p) & k = 0 \end{cases}$$



Example: Indicator function.

Let $A \subset S$

$$I_A(s) = \begin{cases} 1 & s \in A \\ 0 & s \notin A \end{cases}$$



Thm $\qquad I_A \sim \text{Bernoulli}(P(A))$

Geometric RV.

- Number of independent Bernoulli trials until a success is obtained.

$$X \sim \text{Geometric}(p), \quad R_X = \{1, 2, 3, \ldots\} = \mathbb{N}.$$

$$P_X(k) = \begin{cases} p(1-p)^{k-1}, & k = 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

How to see this?

$$\begin{array}{cc} (1-p)^{k-1} & p \\ \uparrow & \uparrow \\ \underbrace{T\,T \ldots T}_{k-1} & \underset{1}{H} \end{array}$$

Question: $\displaystyle\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = p\sum_{k=1}^{\infty}(1-p)^{k-1}$

$$\underset{k-1=j}{=} p \sum_{j=0}^{\infty} (1-p)^j = p\left(\frac{1}{1-(1-p)}\right) = \frac{p}{p} = 1$$

Binomial Distribution.

- Number of successes in n independent Bernoulli trials?,

- How many Heads in n coin flips?

Def

$X \sim \text{Binomial}(n, p)$ if $R_X = \{0, 1, \dots, n\}$

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

How to see this?

Consider $\underbrace{H\,H \dots H}_{k}\ \underbrace{T\,T \dots T}_{n-k}.$

$$\text{prob} \sim p^k (1-p)^{n-k}$$

How many ways to "partition" into k-heads, $n-k$ tails.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Show that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.$

Important way to write Binomial (n, p).

Let $X_1, X_2, \dots X_n$ be independent Bernoulli(p) RV's, then

$$X = X_1 + X_2 + \dots + X_n \sim \text{Binomial}(n, p).$$

$$= \# \text{ of sucesses.}$$

<u>Ex</u> $X \sim \text{Binomial}(n, p)$, $Y \sim \text{Binomial}(m, p)$. independent.

What is $Z = X + Y$?

$$X = X_1 + X_2 + \dots X_n, \quad Y = Y_1 + Y_2 + \dots + Y_m$$

$$Z = \underbrace{(X_1 + X_2 + \dots + X_n) + (Y_1 + Y_2 + \dots + Y_m)}_{n+m}$$

$Z \sim \text{Binomial}(n+m, p)$.

Pascal Distribution. (Negative Binomial.)

- Number of independent Bernoulli trials until m successes.

Def $X \sim \text{Pascal}(m, p)$, $R_X = \{m, m+1, \dots\}$,

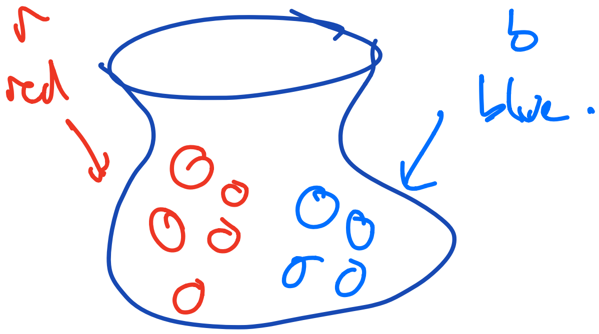
$$P_X(k) = \binom{k-1}{m-1} p^m (1-p)^{k-m} \quad k = m, m+1, \dots$$

How to see?

$$\underbrace{\binom{k-1}{m-1} p^{m-1} (1-p)^{k-m}}_{\text{prob that out of } k-1 \text{ flips you get } m-1 \text{ heads.}} \quad p$$

p: prob that the k^{th} flip is heads.

Hyper-geometric Distribution (Urn problem w/o replacement)



Sample $k \le r + b$ w/o replacement.

$X = \#$ blue.

$$\max(0, k-r) \le X \le \min(k, b)$$

ways to pick x blues

ways to pick $k-x$ red.

$$P_X(x) = \frac{\binom{b}{x}\binom{r}{k-x}}{\binom{b+r}{k}}, \quad x \in R_X$$

$$X \sim \text{Hypergeometric}(b, r, k).$$