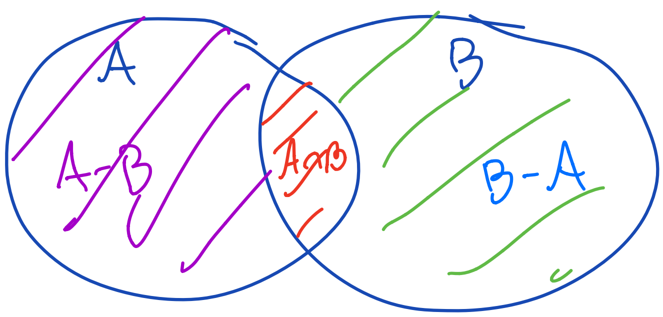
Review 1

Set Theory

S - sample space, ϕ empty set.

- $A \cup B$ - "A or B" - union
- $A \cap B$ - "A and B" - intersection
- A^c - "not A" - complement.
- $A \setminus B = A - B = A \cap B^c$ - "A minus B" - subtraction.
- $A \cap B = \phi$ mutually exclusive / disjoint

Venn Diagram



Laws

De Morgans Laws, Distributive Laws

$$(A \cup B)^c = A^c \cap B^c \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$(A \cap B)^c = A^c \cup B^c \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$|A|$ = cardinality = # elements in A. discrete

infinite

countable $\mathbb{N}$

uncountable $\mathbb{R}$ continuous

Probability Axioms of probability measure.

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

disjoint $A_i \cap A_j = \emptyset \quad i \neq j$

"Laws"

- Complement $P(A^c) = 1 - P(A)$

- Subtraction.

$$P(A) = P(A \setminus B) + P(B \cap A)$$

$$P(A \cup B) = P(A \setminus B) + P(B)$$

- Addition (inclusion exclusion).

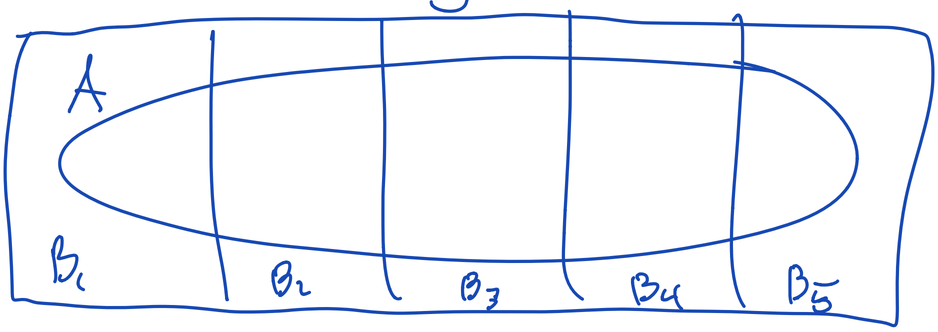
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Conditional probability.

Prob of A given B $\quad P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \; P(B) \neq 0$

Law of total probability.



$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)\,P(B_i)$$

$\{B_i\}$ - mutually exclusive

Bayes Rule

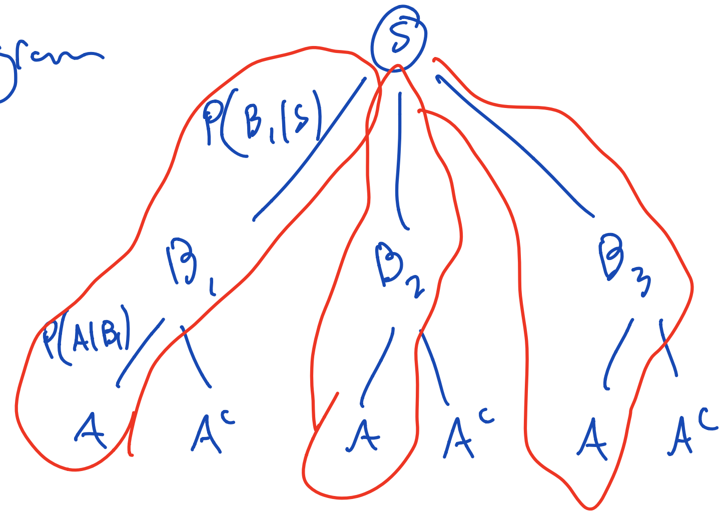
the amount of evidence that A has supporting B.

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)} = \left[\frac{P(A|B)}{P(A)}\right] P(B)$$

Update the probability $P(B)$ given "evidence" A.

$P(A)$ - computed by Law of total prob.

Tree Diagram



$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

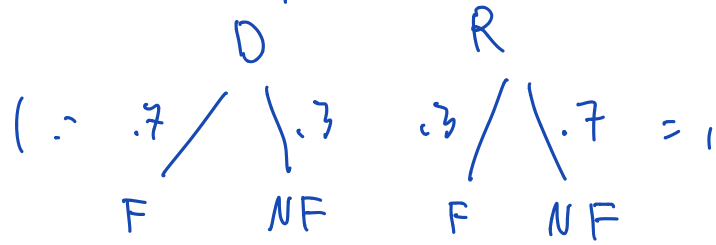
Ex Population. 40% Republican 60% Democrat.

Issue: 30% Republicans in favor
70% Democrats in favor.

Suppose you randomly choose someone. What is the probability they are Democrats, given that they support the issue?

Use Bayes formula

$$\underset{\text{Democrat}}{P(D} \mid \underset{\text{in favor}}{F)} = \frac{P(F|D)P(D)}{P(F)}$$



$$P(F) = .7(.6) + .3(.4) = .54$$

$$P(D \mid F) = \frac{.7(.6)}{.54} = .77\ldots$$

Combinatorics + Sampling.

$|S| < \infty$ finite sample space.

Equally likely outcomes $\boxed{P(A) = \frac{|A|}{|S|}}$

Multiplication Principle

r - experiments each has n_r possible outcomes then there are

$$n_1 \times n_2 \times \ldots \times n_r$$

possible outcomes for each experiment performed in succession.

Sampling

$A = \{1, 2, \ldots, n\}$, $\{2, 3, 4\}$

"Draw" or sample a subset of k elements from it.

How many ways are there to do this?

1. Ordered with replacement. , n^k
2. Ordered without replacement. , $P_k^n = \frac{n!}{(n-k)!}$
   ↳ permutations.
3. Unordered without replacement. $C_k^n = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$
   ↳ combinations
4. Unordered with replacement. $\binom{n+k-1}{k}$

Ex

1. How many 8 digit numbers are there? 10^8
2. How many 8 digit numbers are there, where no two digits are the same? $P_8^{10} = \frac{10!}{2!}$

- Birthday problem.

③ Poker hands, How many ways to draw 10 cards out of a deck?

④ Ex How many solutions are there to

$$n_1 + n_2 + \dots + n_k = n \qquad \binom{n+k-1}{k}$$

$$n_i \in \{0, 1, 2, \dots, n\}.$$

Bars and star method.

$$|||*||*|*|||$$

Ex
How many ways are there to distribute 7 dollar bills between 3 people so that at least each person has 1 dollar.

$$n_1 + n_2 + n_3 = 7 \qquad n_i \in \{1, 2, \dots 7\}$$

$$y_i = n_i - 1 \in \{0, 1, 2 \dots 7\}$$

$$y_1 + y_2 + y_3 - 3 = 7$$

$$\Rightarrow \binom{10 + (3-1)}{3}$$

$$y_1 + y_2 + y_3 = 10$$

$$\overline{\qquad\qquad\qquad\qquad} = \binom{12}{3}$$

Partitions Number of ways to permute n things into k groups of size $n_1, n_2, \ldots, n_k$.

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Ex How many unique ways are there to rearrange a word?

RACECAR.

2-R's, 2-A's, 2-C's, 1-E.
n_1 n_2 n_3 n_4

$$\binom{7}{2,2,2,1} =$$

Interpretation: How many ways are there to put n balls into k bins with n_1 in the first, n_2 balls into the second, ...

Multiplication principle

$$\binom{n}{n_1, n_2, \ldots, n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_k}{n_k}$$

first bin — second bin

Discrete Random Variables

Def: $X : S \to \mathbb{R}$

$R_X = \text{Range}(X)$ all the values X can take.

↳ Discrete if R_X countable.

PMF
$$P_X(x) = P(X = x) \qquad x \in R_X$$

CDF $F_X(x) = P(X \leq x) \qquad x \in \mathbb{R}.$

Expectation.

$$\mathbb{E}X = \sum_{x \in R_X} x \, P_X(x)$$

Linearity.

$$\mathbb{E}\{aX + bY\} = a\,\mathbb{E}X + b\,\mathbb{E}Y.$$

Variance

$$Var(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$Var(cX) = c^2 Var(X) \quad , \quad Var(X+c) = Var(X)$$

Special Distributions

- Bernoulli $\quad X \sim Bernoulli(p) \quad p \in [0,1]$

$$P_X(x) = \begin{cases} p & x = 1 \\ \underset{\overset{\shortparallel}{q}}{1-p} & x = 0 \end{cases} \qquad x \in R_X = \{0, 1\}.$$

$$\mathbb{E}X = p \qquad , \qquad Var(X) = pq.$$

Ex Random coin toss

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise.} \end{cases} \qquad A \subseteq S.$$

- Geometric $\quad X \sim Geometric(p) \quad p \in [0,1].$

$$P_X(x) = (1-p)^{k-1} p \qquad k \in R_X = \{1, 2, 3, \dots\}.$$

$$\mathbb{E}X = 1/p \quad , \quad Var(X) = \frac{1-p}{p^2}.$$

Ex the number of Bernoulli trials until success!

Helps to remember: $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$.

How to compute. $\sum_{k=1}^{\infty} k q^k$?

$$q\left(\sum_{k=0}^{\infty} k q^{k-1}\right) = \frac{q}{(1-q)^2}$$

$$\frac{d}{dq}\sum_{k=0}^{\infty} q^k = \frac{d}{dq}\frac{1}{1-q} = \frac{1}{(1-q)^2}$$

Binomial Distribution. $X \sim \text{Binomial}(n, p)$

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x \in R_X = \{0, \dots, n\}$$

$\mathbb{E}X = np$, $\mathbb{E}X = npq$.

- the number of successes in n consecutive bernoulli trials?

Hyper geometric $X \sim \text{Hypergeometric}(b, r, k)$

$$P_X(x) = \frac{\binom{b}{x}\binom{r}{k-x}}{\binom{b+r}{k}} \qquad x \in R_X.$$

$$\binom{b+r}{k}$$

Don't worry about mean and variance.

$$R_X = \{\max\{0, k-r\}, \dots, \min\{k, b\}\}$$

- Urn (bag) with b - blue balls, r - red balls

What is the probability of getting x blue balls out k total (sampled w/o replacement).

- <u>Pascal Distribution</u> $X \sim \text{Pascal}(m, p)$.

$$X = X_1 + X_2 + \dots + X_m, \quad X \sim \text{Geometric}(p).$$

How trials do you have to wait to get exactly m heads?

$$P_X(x) = \binom{x-1}{m-1} p^m (1-p)^{x-m}, \quad x \in \{m, m+1, \dots\}.$$

$$\mathbb{E}X = \frac{m}{p}, \quad \text{Var}(X) = \frac{m(1-p)}{p}$$

<u>Poisson</u> $X \sim \text{Poisson}(\lambda)$.

$$P_X(x) = \frac{e^{-\lambda}\lambda^x}{\dots}, \quad x \in R_X = \{0, 1, 2, \dots\}.$$

$x!$

$$\mathbb{E}X = \lambda \quad , \quad Var(X) = \lambda .$$

λ - rate of occurance of rare events.

- Limit of Binomial distribution. $n \to \infty$ $p = \lambda/n$.