Review 2

Outline

Continuous Random Variables

Def: X is continuous if it's CDF

$$F_X(x) = P(X \leq x)$$ is continuous an differentiable.

PDF: $f_X(x) = \frac{d}{dx} F_X(x)$.

$$\Rightarrow P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

Expectation + Variance

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx, \quad Var(X) = \mathbb{E}(X - \mathbb{E}X)^2$$

$$= \int_{-\infty}^{\infty} (x - \underset{\underset{\mathbb{E}X}{\parallel}}{\mu_X})^2 f_X(x)\,dx$$

LOTUS

$$\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx.$$

Special Distributions

Uniform (a,b) $\quad X \sim \text{Uniform}(a,b)$.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b]. \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}X = \frac{b+a}{2} \qquad \text{Var}(X) = \frac{(b-a)^2}{12}$$

midpoint

Exponential (λ) $\quad X \sim \text{Exponential}(\lambda)$.

rate.

- Random alarm clock

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}X = 1/\lambda \quad , \quad \text{Var}(X) = 1/\lambda^2$$

$\hookrightarrow$ integration by parts.

Normal (μ, σ^2) , $\quad X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x-\mu|^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

$\mathbb{E}X = \mu \quad , \quad Var(X) = \sigma^2.$

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1) \quad , \quad X = \sigma Z + \mu.$$

↑ standardize.

<u>Gamma</u> (α, λ) ← rate.

$$X \sim Gamma(\alpha, \lambda)$$

$$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$

Gamma func

↳ extension of the factorial to real numbers.

$$\mathbb{E}X = \alpha/\lambda \quad , \quad Var(X) = \frac{\alpha}{\lambda^2}$$

<u>Properties</u>

① Exponential (λ) = Gamma $(1, \lambda)$

② $X_1 + X_2 + \dots + X_n = X \sim Gamma(n\alpha, \lambda)$

Gamma (α, λ)

Functions of RVs.

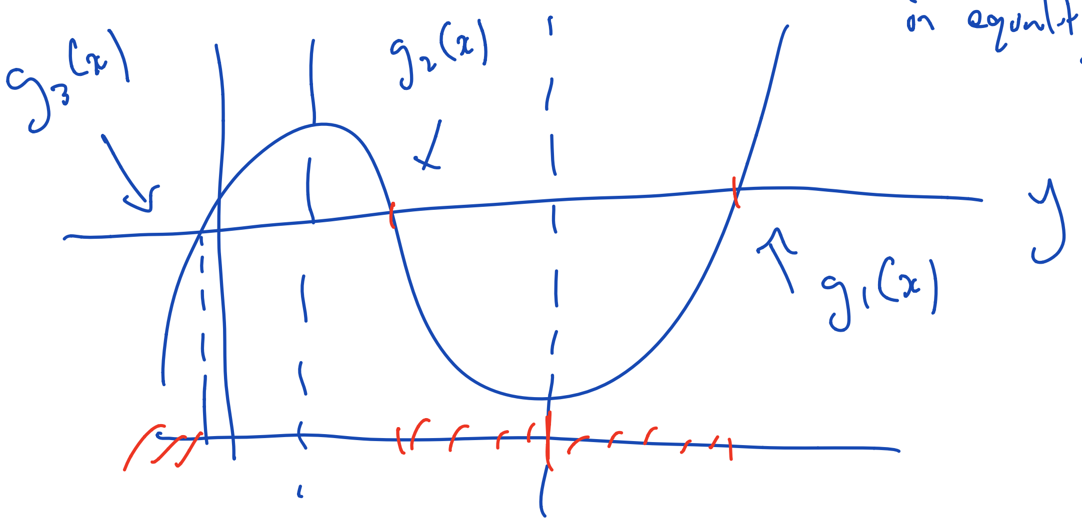
$X \sim$ PDF $f_X(x)$.

What is the distribution of $Y = g(X)$?

CDF Method.

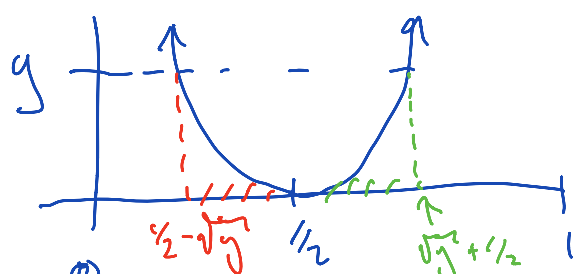
① $F_Y(y) = P(Y \le y) = P(g(X) \le y)$

↑ manipulate this inequality.



Ex $X \sim$ Uniform $(0, 1)$ $\qquad g(x) = (x - 1/2)^2$

$Y = g(X)$?



$$F_Y(y) = P(Y \le y) = P\left((X - 1/2)^2 \le y\right)$$

$$(X - 1/2)^2 \le y \iff \begin{cases} (X - 1/2) \le \sqrt{y} & \text{green} \\ -(X - 1/2) \le \sqrt{y} & \text{red.} \end{cases}$$

$$F_Y(y) = P\left(-\sqrt{y} \le (X - 1/2) \le \sqrt{y}\right)$$

$$= P\left(X - 1/2 \le \sqrt{y}\right) - P\left(X - 1/2 \le -\sqrt{y}\right)$$

$$= P\left(X \le \sqrt{y} + 1/2\right) - P\left(X \le 1/2 - \sqrt{y}\right)$$

$$= F_X\left(\sqrt{y} + 1/2\right) - F_X\left(1/2 - \sqrt{y}\right).$$

Take derivative.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2\sqrt{y}} f_X\left(\sqrt{y} + 1/2\right) + \frac{1}{2\sqrt{y}} f_X\left(1/2 - \sqrt{y}\right)$$

$$= \begin{cases} \frac{1}{2\sqrt{y}} & \sqrt{y} + 1/2 \in [0,1] \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \frac{1}{2\sqrt{y}} & 1/2 - \sqrt{y} \in [0,1]. \\ 0 & \text{otherwise.} \end{cases}$$

$$\sqrt{y} + 1/2 \in [0,1] \iff \sqrt{y} \in [-1/2, 1/2]$$

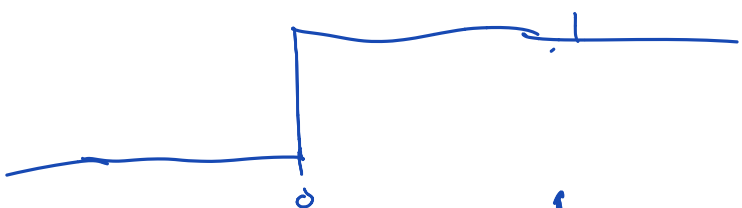
$$\iff y \in [0, 1/4]$$

$$f_X(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & y \in [0, 1/4] \\ 0 & \text{otherwise} \end{cases}$$

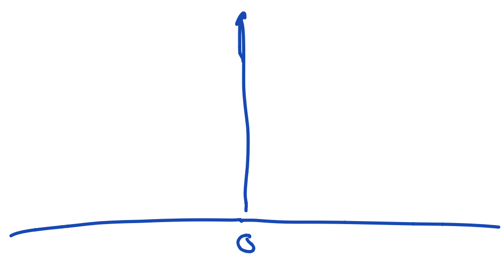
Mixed RVs.

CDF has jumps. (Discrete / Continuous)

$$u(x) = \mathbb{I}_{[0,\infty)}.$$

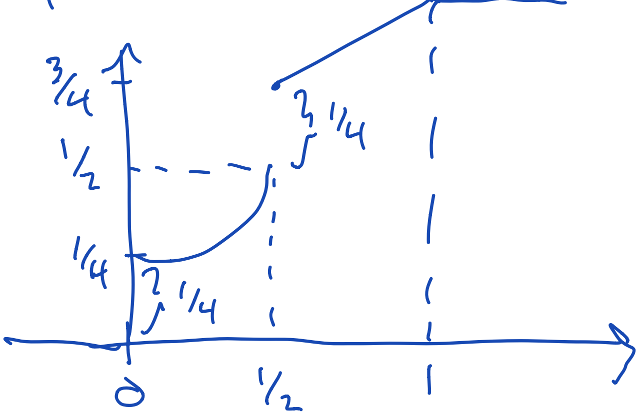


$$\delta(x) = \frac{d}{dx} u(x).$$

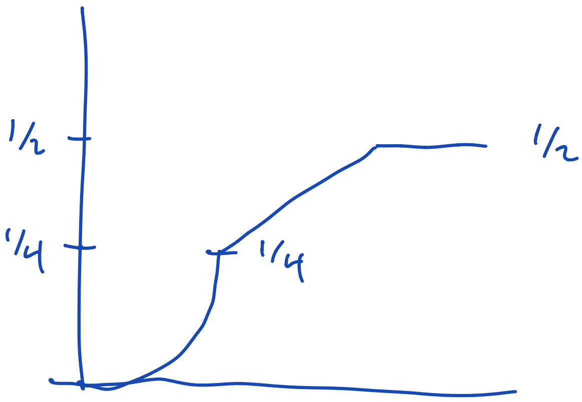


Ex

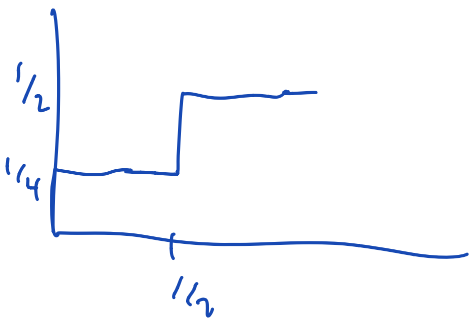
$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^2 + 1/4 & 0 \le y < 1/2 \\ 1/2\, y + 1/2 & 1/2 \le y < 1. \\ 1 & y \ge 1 \end{cases}$$



$$F_Y(y) = C(y) + D(y)$$



$$C(x) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \le y < 1/2 \\ 1/2\, y & 1/2 \le y < 1 \\ 1/2 & y \ge 1 \end{cases}$$



$$D(y) = 1/4\, u(y) + 1/4\, u(y - 1/2)$$

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1/2 \\ 1/2 & 1/2 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} + 1/4\, \delta(y) + 1/4\, \delta(y - 1/2)$$

Property

$$\int_{-\infty}^{\infty} g(x)\, \delta(x - x_0)\, dx = g(x_0)$$

Multivariate probability.

Continuous.

$$
\begin{tikzcd}
 & & & f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} \\
 & & f_X(x) \arrow[ur] \arrow[dr, "\text{indep}"] & \\
F_{XY}(x,y) \arrow[r, "\frac{\partial}{\partial x}\frac{\partial}{\partial y}"] & f_{X,Y}(x,y) \arrow[ur, "\int \cdot\, dy"] \arrow[dr, "\int \cdot\, dx"'] & & f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \\
 & & f_Y(y) \arrow[ur, "\text{indep}"'] \arrow[dr] & \\
 & & & f_{Y|X}(x|y) = \dfrac{f_{XY}(x,y)}{f_X(x)}
\end{tikzcd}
$$

LoTUS.

$$\mathbb{E}\, g(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy.$$

Conditional Expectation / Variance.

$$\mathbb{E}\{X \mid Y=y\} = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx = \mu_{X|Y}(y)$$

$$\hookrightarrow \quad \mathbb{E}\{X \mid Y\} = \mu_{X|Y}(Y). \quad \checkmark \text{ Now an RV.}$$

$$\mathrm{Var}(X \mid Y=y) = \int^{\infty} (x - \mu_{X|Y}(y))^2\, f_{X|Y}(x|y)\, dx$$

$\hookrightarrow \mathrm{Var}(X \mid Y) = \sigma^2_{X|Y}(Y)$

Law of total Expectation on Variance.

$$\mathbb{E}X = \mathbb{E}\{\mathbb{E}[X \mid Y]\}$$

$$\mathrm{Var}(X) = \mathbb{E}\{\mathrm{Var}(X \mid Y)\} + \mathrm{Var}(\mathbb{E}(X \mid Y)).$$

Random

Ex $\qquad X = \sum_{i=1}^{N} X_i$

Covariance + Correlation.

$$\mathrm{Cov}(X,Y) = \mathbb{E}(X - \mu_X)(Y - \mu_Y)$$

$$= \mathbb{E}(XY) - \mu_X \mu_Y.$$

- Measures the linear dependence of X, Y.

$$\begin{cases} \text{Independent} \Rightarrow \mathrm{Cov} = 0 \\ \mathrm{Cov} = 0 \not\Rightarrow \text{Independent}. \end{cases}$$

Ex $f_{XY} = \begin{cases} c & -1 \le x \le 1,\ 0 \le y \le x^2 \\ 0 & \text{otherwise.} \end{cases}$

Correlation.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \, \sigma_Y}, \quad -1 \le \rho(X,Y) \le 1$$

$$Cov(X+Y, Z) = Cov(X,Z) + Cov(Y,Z).$$

$$Cov(X,X) = Var(X).$$

Ex Let X be the score of a random student on the final. Let Y be the number of hours they studied.

$$X = 2Y + Z, \quad Z \text{ independent of } Y$$

$$Var(X) = 40, \quad Var(Y) = 2$$

a) $Var(Z)$

b) $Cov(X,Y)$

a) $Var(X) = Var(2Y + Z)$

$= Var(2Y) + Var(Z)$

$= 4Var(Y) + Var(Z).$

$Var(Z) = Var(X) - 4Var(Y)$

$= 40 - 4(2) = 32$

b) $Cov(X, Y) = Cov(2Y + Z, Y)$

$= 2\underbrace{Cov(Y, Y)}_{= Var(Y).} + \underbrace{Cov(Z, Y)}_{= 0}.$

$= 2Var(Y) = 4.$

$\rho(X, Y) = \frac{4}{\sqrt{40}\sqrt{2}} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}}.$

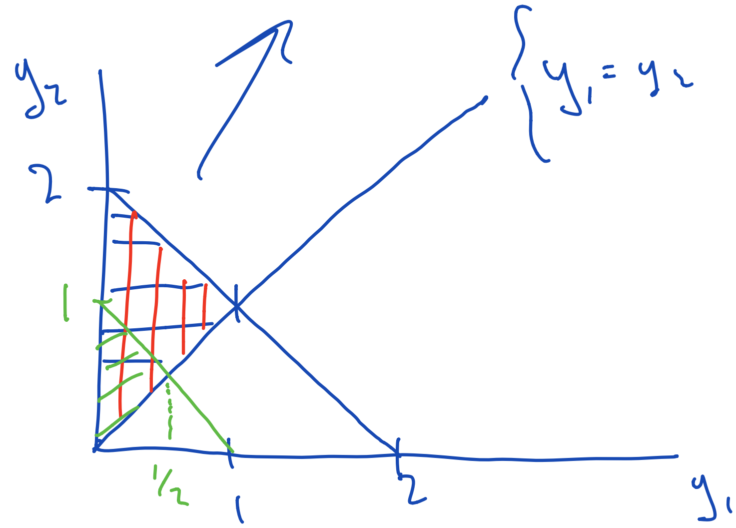
Ex

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6y_1^2 y_2 & 0 \le y_1 \le y_2 \\ & y_1 + y_2 \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

① Is this a valid PDF.

② $P(Y_1 + Y_2 < 1)$

③ Are Y_1, Y_2 independent?

① 

$\{y_1 = y_2, \quad y_2 \le 2 - y_1\}$,

$$R_{Y_1,Y_2} = \{(y_1, y_2) : \quad 0 \le y_1 \le 1, \; y_1 \le y_2 \le 2 - y_1\}.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_1 dy_2 = \int_0^1 \int_{y_1}^{2-y_1} 6y_1^2 y_2 \, dy_2 \, dy_1$$

$$= \int_0^1 6y_1^2 \int_{y_1}^{2-y_1} y_2 \, dy_2 \, dy_1$$

$$\tfrac{1}{2}(2-y_1)^2 - \tfrac{1}{2}y_1^2$$
$$= \tfrac{1}{2}2^2 - \tfrac{1}{2}4y_1$$
$$= 2(1-y_1)$$

$$= \int_0^1 6y_1^2 \left\{ \tfrac{1}{2} y_2^2 \Big|_{y_1}^{2-y_1} \right\} dy_1$$

$$= \int_0^1 12\, y_1^2 (1-y_1)\, dy_1 = 12\left(\tfrac{1}{3} y_1^3 \Big|_0^1 - \tfrac{1}{4} y_1^4 \Big|_0^1\right)$$

$$= 12\left(\tfrac{1}{3} - \tfrac{1}{4}\right) = \frac{12}{12} = 1.$$

$\int_0^{1/2} \int^{1-y_1}$

② $P(Y_1 + Y_2 < 1) = \int_0 \int_{y_1} 6 y_1^2 y_2 \, dy_2 \, dy_1$

$$= \int_0^{1/2} 6 y_1^2 \int_{y_1}^{1-y_1} y_2 \, dy_2 \, dy_1$$

$$= \int_0^{1/2} 6 y_1^2 \left| \tfrac{1}{2} y_2^2 \right|_{y_1}^{1-y_1} dy_1$$

$$\tfrac{1}{2}(1-y_1)^2 - \tfrac{1}{2} y_1^2$$

$$= \tfrac{1}{2} - y_1$$

$$= \int_0^{1/2} 6 y_1^2 \left(\tfrac{1}{2} - y_1\right) dy_1 = 3\left[\tfrac{1}{3} y_1^3 \Big|_0^{1/2}\right] - 6\left[\tfrac{1}{4} y_1^4 \Big|_0^{1/2}\right]$$

$$= \left(\tfrac{1}{2}\right)^3 - \tfrac{3}{2}\left(\tfrac{1}{2}\right)^4$$

$$= \left(\tfrac{1}{2}\right)^3 \left[1 - \tfrac{3}{4}\right]$$

$$= \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{4}\right) = \frac{1}{32}$$?

③ <u>Not independent</u>.

$$f_{X_2}(y_2) = \begin{cases} \int_0^{y_2} 6 y_1^2 y_2 \, dy_1 & \text{if } y_2 \in [0,1] \\ \int_0^{2-y_2} 6 y_1^2 y_2 \, dy_1 & \text{if } y_2 \in [1,2]. \end{cases}$$

$$= \begin{cases} 2 y_2^4 & y_2 \in [0,1] \\ 2(2-y_2)^2 y_2 & y_2 \in [1,2]. \end{cases}$$

$$f_{Y_1}(y_1) = \int_{y_1}^{2-y_1} 6y_1^2 y_2 \, dy_2 = 3y_1^2 \left[y_2^2 \Big|_{y_1}^{2-y_1} \right]$$

$$= 12y_1^2(1-y_1)$$

$$f_{Y_1}(y_1) = \begin{cases} 12y_1^2(1-y_1) & y_1 \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{Y_2}(y_2) = \begin{cases} 2y_2^4 & y_2 \in [0,1] \\ 2(2-y_2)^2 y_2 & y_2 \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

$\hookrightarrow$ Not independent ...