Take-Home Exam - Solutions

Problem 1 (Continuous Dependence): Let $U \in \mathbb{R}^n$ be a bounded domain. Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad U \\ u = g & \text{on} \quad \partial U. \end{cases}$$

Assume f can be continuosly extended to ∂U (i.e. it is uniformly continuous).

(a) Show that the solution u has continuous dependence in $C(\overline{U})$ on the boundary data $g \in C(\partial U)$ and the source term $f \in C(\overline{U})$. Assume that a solution u always exists. Namely, show that if $g_n \to g$ in $C(\partial U)$ and $f_n \to f$ in $C(\overline{U})$ that the solution $u_n \to u$ in $C(\overline{U})$.

(b) Show that the solution has continuous dependence in $C^k(\overline{U})$ on the boundary data $g \in C^k(\partial U)$ and the source term $f \in C^k(\overline{U})$ for any $k \geq 1$ (again assume f and its first k derivatives can be continuously extended to $\overline{U}$).

Solution:

(a) We follow one of the proofs in the homework (repeated here for convenience). Namely let $v = u - \lambda|x|^2$, we know that

$$-\Delta v = f - \lambda 2n \leq 0$$

if $\lambda = \|f\|_{C(U)}/2n$. Therefore $v = u - \lambda|x|^2$ is a subsolution and so by the maximum principle and the fact that $v \leq g$ on ∂U, we find $\max_{\overline{U}} v \leq \max_{\partial U} g$. This implies that

$$\max_{\overline{U}} u \leq \max_{\partial U} g + \frac{\|f\|_{C(U)}}{2n} \min_{U} |x|^2.$$

Applying the same argument to $-u$ gives

$$\max_{\overline{U}} -u \leq \max_{\partial U} -g + \frac{\|f\|_{C(U)}}{2n} \min_{U} |x|^2.$$

Putting these together gives the estimate

$$\|u\|_{C(\overline{U})} \leq C(\|f\|_{C(\overline{U})} + \|g\|_{C(\partial U)}),$$

where the constant C doesn't depend on u. Now let $f_n \to f$ in $C(U)$ and $g_n \to g$ in $C(\partial U)$, then by linearity we have that the associated solutions u_n and u satisfy

$$\|u - u_n\|_{C(\overline{U})} \leq C(\|f - f_n\|_{C(\overline{U})} + \|g_n - g\|_{C(\partial U)}).$$

Sending $n \to \infty$ gives the result.

(b) To get C^k, just take the derivative of the equation and apply the same argument to $D^\alpha u$ for $|\alpha| = k$.

Problem 2 (Greens Function): Let $U \subseteq \mathbb{R}^n$ have C^1 boundary (but is not necessarily bounded). Recall, the Neumann boundary value problem takes the form

$$\begin{cases} \Delta u = 0 & \text{in} \quad U \\ \frac{\partial u}{\partial \nu} = f & \text{on} \quad \partial U, \end{cases}$$

for some function $f \in C(\partial U)$ with $\int f \mathrm{d}S(y) = 0$, where ν denotes the outward facing normal to ∂U. Note that $\int_{\partial U} f \mathrm{d}S(y)$ is necessary since by the divergence theorem

$$\int_{\partial U} f(y) \mathrm{d}S(y) = \int_{\partial U} \partial_\nu u(y) \mathrm{d}S(y) = \int_U \Delta u \, \mathrm{d}y = 0$$

(a) (Extra Credit) We seek a Green's function $G(x, y)$ that can express a solution $u \in C^2(\overline{U})$ as

$$u(x) = c + \int_{\partial U} G(x, y) f(y) \mathrm{d}S(y),$$

where f is assumed to have compact support on ∂U, $\int_{\partial U} f \mathrm{d}S(y) = 0$ and c is a constant that depends on u. Give a formula for the Green's function in terms of a corrector $\phi^x(y)$ that is chosen to satisfy a particular BVP. (Hint, you may take for granted the formula we proved in class

$$u(x) = \int_{\partial U} \partial_\nu \Phi(x - y) u(y) - \partial_\nu u(y) \Phi(x - y) \mathrm{d}S(y),$$

where Φ is the fundamental solution.)

(b) Consider the Neumann problem in the upper half-plane $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$,

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^2_+ \\ u_{x_2} = f & \text{on} \quad \{x_2 = 0\}. \end{cases}$$

Find the corresponding Green's function and show that

$$u(x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}} \ln[(x_1 - y)^2 + x_2^2] \, f(y) \, \mathrm{d}y.$$

is a solution.

Solution:

(a) We will seek a corrector $\phi^x(y)$ that solves the following problem

$$\begin{cases} \Delta\phi^x(y) = 0 & \text{in } U \\ \partial_\nu \phi^x(y) = \partial_\nu \Phi(x-y) - h(y) & \text{on } \partial U \end{cases}$$

for some function $h(y)$ on ∂U with $\int_{\partial U} h \mathrm{d}x = 1$. Note we must include the extra function h in the boundary condition since the Neumann problem must have mean zero boundary condition by the divergence theorem

$$0 = \int_U \Delta\phi^x \mathrm{d}y = \int_{\partial U} \partial_\nu \phi^x(y) \mathrm{d}S(y).$$

Since we know that (at least for bounded U) $\int_{\partial U} \partial_\nu \Phi(x-y)\mathrm{d}S = \int_U \Delta_y \Phi(x-y)\mathrm{d}y = 1$, this means that $\int_{\partial U} h \mathrm{d}S = 1$. Applying Green's formula, $\phi^x(y)$ satisfies

$$\int_{\partial U} \phi^x(y)\partial_\nu u(y)\mathrm{d}S(y) = \int_{\partial U} \partial_\nu \Phi(x-y)u(y) - h(y)u(y)\mathrm{d}S(y) = 0.$$

Defining the Green's function $G(x,y) = -\Phi(x-y) + \phi^y(x)$ and using that $\partial_\nu u(y) = f(y)$ on ∂U we can combine this with the formula proved in class

$$u(x) = \int_{\partial U} \partial_\nu \Phi(x-y)u(y) - \partial_\nu u(y)\Phi(x-y)\mathrm{d}S(y)$$

to obtain

$$u(x) = \int_{\partial U} h(y)u(y)\mathrm{d}S(y) + \int_{\partial U} G(x,y)f(y)\mathrm{d}S(y).$$

Defining $c = \int_{\partial U} h(y)u(y)\mathrm{d}S(y)$ gives the formula.

(b) Lets use the corrector approach to solve the Neumann problem in $\mathbb{R}^2_+$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+ \\ u_{x_2} = f & \text{on } \{x_2 = 0\}. \end{cases}$$

Note, the problem above is stated in terms of the inward normal $u_{x_2} = -\partial_\nu u = f$, so we must carry around the minus sign. To find the solution, we must solve the corrector problem we posed in part a). To do this, we use a reflection principle. Namely let $\tilde{x} = (x_1, -x_2)$ and define

$$\phi^x(y) = -\Phi(y - \tilde{x}) = -\frac{1}{4\pi}\ln((y_1 - x_1)^2 + (y_2 + x_2)^2).$$

Note that clearly $\Delta\phi^x(y) = 0$ since we moved the singularity away from the inside of $\mathbb{R}^2_+$. Moreover

$$\partial_{y_2}\phi^x(y)|_{y_2=0} = \frac{-1}{2\pi}\frac{x_2}{(y_1-x_1)^2 + x_2^2} = \partial_{y_2}\Phi(y-x)|_{y_2=0}.$$

Note that we don't need the h here since U is unbounded. It follows that the Green's function is

$$G(x,y) = -\Phi(x-y) - \Phi(y-\tilde{x}).$$

This gives the formula (keeping in mind ∂_{y_2} is the inward facing normal derivative)

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \ln\left((y_1-x_1)^2+x_2^2\right) f(y_1)\mathrm{d}y_1.$$

Now lets justify that this formula actually satisfies the Neumann problem. We will assume that f is continuous with compact support. To show this, we note first that for each y and $(x_1,x_2)\in\mathbb{R}^2_+$, $\Delta\ln\left((y-x_1)^2+x_2^2\right) = 0$ since we are away from the singularity at $x_2=0$. Therefore, if we are in the regime where f has compact support, then we can pull the Laplacian inside the integral and conclude that

$$\Delta u(x_1,x_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \Delta\ln\left((y-x_1)^2+x_2^2\right) f(y)\mathrm{d}y = 0.$$

To see the boundary condition is met, we note that

$$\partial_{x_2}u(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x_2}{(y-x_1)^2+x_2^2} f(y)\mathrm{d}y$$

Since

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{x_2}{(y-x_1)^2+x_2^2}\mathrm{d}y = \frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{y^2+1}\mathrm{d}y = 1,$$

we have

$$\begin{aligned}|\partial_{x_2}u(x_1,x_2) - f(x_1)| &\le \frac{1}{\pi}\int_{\mathbb{R}} \frac{x_2}{(y-x_1)^2+x_2^2}|f(y)-f(x_1)|\mathrm{d}y \\ &= \frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{y^2+1}|f(x_2y+x_1)-f(x_1)|\mathrm{d}y\end{aligned}$$

where we changed variables to obtain the last line. If f is assumed to be continuous, we note that $\lim_{x_2\to 0^+} f(x_2y+x_1) = f(x_1)$ for each y. Therefore since f is also bounded and $\frac{1}{1+y^2}$ is integrable we can use dominated convergence to conclude that

$$\lim_{x_2\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{y^2+1}|f(x_2y+x_1)-f(x_1)|\mathrm{d}y = 0$$

and therefore $\lim_{x_2\to 0^+}\partial_{x_2}u(x_1,x_2) = f(x_1)$.

Problem 3 (Soap bubble). Let U be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and let φ be a smooth function on $\overline{U}$ satisfying $\varphi|_{\partial U} < 0$. Define the constraint set

$$\mathcal{A}_\varphi := \{u\in C^2(\overline{U})\,:\, u|_{\partial U} = 0,\, u\ge\varphi \,\text{on}\, \overline{U}\}$$

and the surface energy of u

$$I[u] := \frac{1}{2}\int_U |Du(x)|^2 \mathrm{d}x.$$

We seek to minimize the surface energy $I[u]$ over all functions $u \in \mathcal{A}_\varphi$ constrained to lie above φ. Physically, we can view the minimizer as the shape of a “soap bubble” attached to the boundary ∂U and stretched over the obstacle φ. The soap bubble will try to minimize its energy as best it can subject to constraint imposed by the “obstacle”.

Show that if such a minimizer exists, i.e. if there is a $u_* \in \mathcal{A}_\varphi$ such that

$$I[u_*] = \min\{I[u] \,:\, u \in \mathcal{A}_\varphi\},$$

then u_* is harmonic on the open set $V_\varphi := \{x \in U \,:\, u^* > \varphi\}$, namely u_* is harmonic wherever it doesn't touch φ. (Hint: for each test function ϕ compactly supported in $\{u_* > \varphi\}$, show that you can find and interval $(-\delta, \delta)$ of 0, depending on ϕ such that $u_* + \tau\phi > \varphi$ for all $\tau \in (-\delta, \delta)$.)

Solution: Let u_* be a minimizer, $I[u_*] = \min\{I[u] \,:\, u \in \mathcal{A}_\varphi\}$, and fix $\phi \in C_c^\infty(U)$ with $\operatorname{supp}\phi \subseteq V_\varphi$. We want to show that $u^* + \tau\phi$ still belongs to the constraint set $\mathcal{A}_\varphi$ for τ in a suitably small neighborhood $(-\delta, \delta)$ of 0.

To show this, assume that $\phi \neq 0$ (otherwise we are done) and define for each $n \geq 1$ the open set

$$V_\varphi^n = \{x \in U \,:\, u^* > \varphi + 1/n\}.$$

Clearly we have $V_\varphi = \bigcup_{n=1}^\infty V_\varphi^n$ and therefore $\{V_\varphi^n\}_{n=1}^\infty$ is an open cover of $\operatorname{supp}\phi$. Moreover since $V_\varphi^{n_1} \subseteq V_\varphi^{n_2}$ for $n_1 \leq n_2$, we see that since $\operatorname{supp}\phi$ is compact, there exists an N_ϕ such that $\operatorname{supp}\phi \subset V_\varphi^{N_\phi}$. With this in hand, it is clear that

$$u^* + \tau\phi \geq \varphi$$

as long as $|\tau| \leq \frac{1}{N_\phi \|\phi\|_{L^\infty}}$. It follows that for such τ, $u^* + \tau\phi \in \mathcal{A}_\varphi$. Therefore the function

$$i[\tau] = I[u_* + \tau\phi] = \int_U |Du_*|^2 \,\mathrm{d}x + 2\tau \int_U Du_* \cdot D\phi \,\mathrm{d}x + \tau^2 \int_U |D\phi|^2 \,\mathrm{d}x$$

is $C^1(-\delta, \delta)$ and has a minimum at $\tau = 0$. Necessarily we must have $i'(\tau)|_{\tau=0} = 0$, and so

$$\int_U Du_* \cdot D\phi \,\mathrm{d}x = 0.$$

Integrating by parts and using that ϕ has compact support gives

$$\int_U \Delta u_* \phi \,\mathrm{d}x = 0.$$

Since this holds for each $\phi \in C_c^\infty(U)$ with $\operatorname{supp} V_\varphi = \{u^* > \varphi\}$, we conclude that $\Delta u_* = 0$ on V_φ.

Problem 4 (Regularization and decay): Let u be the solution of Cauchy problem for the heat equation on $\mathbb{R}^n$

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u = f & \text{on} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

where $f \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p \in [1,\infty]$.Show that for each $r \geq p$ and multi-index α, there exist a constant $C \geq 0$ (independent of u) such that

$$\|D^\alpha u(\cdot,t)\|_{L^r} \leq \frac{C}{t^{\frac{|\alpha|}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}} \|f\|_{L^p}$$

(Hint 1: You may want to use Young's convolution inequality $\|f \star g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$), where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.)
(Hint 2: Let $\phi(x) = \exp(-|x|^2/4)$. You may find it useful to show by induction that for any multi-index α there exists a polynomial $P(x)$ (whose exact form isn't important)

$$D^\alpha(\phi(x/\sqrt{t})) = \frac{1}{t^{|\alpha|/2}} P(x/\sqrt{t})\phi(x/\sqrt{t}).$$

You may also find it useful to show that for any polynomial $P(y)$, there is a constant C such that

$$|P(y)\phi(y)| \leq C\phi(y/2).$$

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Solution: Recall the solution to the heat equation can be given by

$$u(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \phi((x-y)/\sqrt{t}) f(y) \mathrm{d}y.$$

Following the hint, we first claim that

$$D^\alpha(\phi(x/\sqrt{t})) = \frac{1}{t^{|\alpha|/2}} P(x/\sqrt{t})\phi(x/\sqrt{t}) \leq \frac{C}{t^{|\alpha|/2}}\phi(x/2\sqrt{t}).$$

Note the second inequality above follows immediately from the fact that $P(x)\phi(x) \leq C$ for some constant C and therefore splitting the exponential

$$P(x)\phi(x) = P(x)\phi(x/2)\phi(x/2) \leq C\phi(x/2).$$

The rest of the claim follows by induction, namely if we assume the formula holds for some multiindex α, then for some other multiindex $\alpha' = \gamma + \alpha$ with $|\gamma| = 1$ and $D^\gamma = \partial_{x_k}$, we find that

$$\begin{aligned} D^{\alpha'}\phi(x/\sqrt{t}) &= \frac{1}{|t|^{|\alpha|/2}} D^\gamma(P(x/\sqrt{t})\phi(x/\sqrt{t})) \\ &= \frac{1}{|t|^{|\alpha|/2+1/2}} \left(\partial_{x_k} P(x/\sqrt{t}) + \frac{2x_k}{\sqrt{t}} P(x/\sqrt{t})\right)\phi(x/\sqrt{t}) \end{aligned}$$

since $P'(x) = \partial_{x_k} P(x) + 2xP(x)$ is another polynomial, this proves the claim.

Using the claim, we find

$$\begin{aligned}|D^\alpha u(x)| &\leq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |D^\alpha(\phi((x-y)/\sqrt{t}))||f(y)|\mathrm{d}y \\ &\leq C\frac{1}{t^{n/2+|\alpha|/2}} \int_{\mathbb{R}^n} \phi((x-y)/2\sqrt{t})|f(y)|\mathrm{d}y.\end{aligned}$$

Applying Young's inequality

$$\left\| \int_{\mathbb{R}^n} \phi((\cdot - y)/2\sqrt{t})|f(y)|\mathrm{d}y \right\|_{L^r} \leq \|\phi(\cdot/2\sqrt{t})\|_{L^q} \|f\|_{L^p}$$

where $\frac{1}{q} = 1 + \frac{1}{r} - \frac{1}{p}$. We see by change of variables that

$$\|\phi(\cdot/2\sqrt{t})\|_{L^q} \leq Ct^{n/2q} = Ct^{\frac{n}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)},$$

where the contant is just a Gaussian integral

$$C = \left(\int_{\mathbb{R}^n} e^{-q|x|^2/4}\mathrm{d}x\right)^{1/q} < \infty.$$

Therefore putting everything together

$$\|D^\alpha u\|_{L^r} \leq \frac{C}{t^{\frac{|\alpha|}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}} \|f\|_{L^p}.$$

Problem 5 (Wave Energy): Use an energy method to show that there is at most one smooth solution to the equation

$$u_{tt} + cu_t - u_{xx} = f$$

in the domain $(0,1) \times (0,\infty)$ with initial conditions $u(x,0) = g(x)$, $u_t(x,0) = h(x)$ and boundary conditions $u(0,t) = u(1,t) = 0$. (Hint: there are different energies for $c > 0$ and $c < 0$)

Solution: We begin by assuming two solutions u, v exist and define $w = u - v$. Therefore w solves the following problem

$$\begin{cases} w_{tt} + cw_t - w_{xx} = 0 & \text{in } (0,1) \times (0,\infty) \\ w = 0, w_t = 0 & \text{at } (0,1) \times \{t = 0\} \\ w = w = 0 & \text{on } \{x = 0, 1\} \times (0,\infty) \end{cases}$$

We start by assuming that $c \geq 0$ and use the energy method. We consider the following integral

$$\int_0^1 (w_{tt} + cw_t - w_{xx})w_t dx = 0$$

Using integration by parts on the last term, we obtain

$$\int_0^1 \frac{1}{2}(w_t)_t^2 + \frac{1}{2}(w_x)_t^2 + cw_t^2 dx + w_x w_t \Big|_0^1 = 0 \tag{1}$$

The boundary terms vanish, since $w(0,t) = w(1,t) = 0$ and hence $w_t(0,t) = w_t(1,t) = 0$. Define the energy of the PDE to be

$$e(t) = \frac{1}{2}\int_0^1 w_t^2 + w_x^2 dx$$

we note that equation (1) implies that

$$e(t)' = -\int_0^1 cw_t^2 dx$$

and therefore $e(t)' \leq 0$, since $c \geq 0$. This gives us that

$$0 \leq e(t) \leq e(0) = \frac{1}{2}\int_0^1 w_t(x,0)^2 + w_x(x,0)^2 dx = 0,$$

since $w(x,0), w_x(x,0)$ are both 0. Therefore $e(t) = 0$ and so $w_t = w_x = 0$. It follows that $w = 0$ in $(0,1)\times(0,\infty)$ since $w(x,0) = w(0,t) = 0$ for all $(x,t) \in (0,1)\times(0,\infty)$, and hence we have uniqueness for $c \geq 0$.

We now consider the case when $c < 0$. Following the steps from before, we obtain

$$\int_0^1 \frac{1}{2}(w_t)_t^2 + \frac{1}{2}(w_x)_t^2 + cw_t^2 dx = 0.$$

Denoting

$$\eta(t) = \frac{1}{2}\int_0^1 w_t^2 dx \quad , \frac{1}{2}\beta(t) = \int_0^1 w_x^2 dx$$

we can write this equation as

$$\eta' + 2c\eta + \beta' = 0.$$

which is just an ODE in η, with $\eta(0) = 0$. Multiplying through by the integration factor e^{2ct}, we obtain

$$(e^{2ct}\eta)' + e^{2ct}\beta' = 0$$

which can be integrated to give

$$\eta(t) = -\int_0^t e^{-2c(t-s)}\beta'(s)ds.$$

Now an integration by parts in the integral gives

$$\eta(t) = -\beta(t) + 2c\int_0^t e^{-2c(t-s)}\beta(s)ds \tag{2}$$

where we have used the fact that $\beta(0) = 0$. This suggests that we want an energy of the form

$$e(t) = \frac{1}{2}\int_0^1 \left[w_t^2 + w_x^2 - 2c\int_0^t e^{-2c(t-s)} w_x(x,s)^2 ds \right] dx.$$

We know from equation (2) that $e(t) = 0$. Therefore since $c < 0$, this implies that $w_t = w_x = 0$ in $(0,1)\times(0,\infty)$, and so by the boundary conditions, $w = 0$ in $(0,1)\times(0,\infty)$. Hence we have uniqueness for $c < 0$. ■

Problem 6 (Wave decay): Let u be a C^2 solution of $u_{tt} - \Delta u = 0$ in $\mathbb{R}^2 \times (0,\infty)$, with

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}^2$$

where g is a smooth function satisfying $g(x) = 0$ for $|x| > a$.

(a) Show that there exists a constant C such that $|u(x,t)| \le \frac{C}{t}$ for $t \ge 2(|x| + a)$

(b) Show that $\lim_{t\to\infty} tu(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} g(y)dy$ for all $x \in \mathbb{R}^2$.

Solution:

(a) Consider Poisson's formula for the solution to wave equation IVP,

$$u(x,t) = \frac{1}{2\pi}\int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} dy.$$

Since $g(x) = 0$ for $|x| > a$, we only need to consider the integration over $B(0,a) \cap B(x,t)$. However, under the assumption that $t \ge 2(|x| + a)$, $B(0,a) \subseteq B(x,r)$. Therefore we consider the integral only over $B(0,a)$

$$u(x,t) = \frac{1}{2\pi t}\int_{B(0,a)} \frac{g(y)}{(1 - (|y-x|/t)^2)^{1/2}} dy.$$

Moreover, whenever $y \in B(0,a)$ and $t \ge 2(|x| + a)$,

$$\frac{|y-x|}{t} \le \frac{|x| + a}{t} \le \frac{1}{2}.$$

Therefore we may apply the following bound to $u(x,t)$

$$\begin{aligned} |u(x,t)| &\le \frac{1}{2\pi t}\int_{B(0,a)} \frac{|g(y)|}{(1 - (|y-x|/t)^2)^{1/2}} dy \\ &\le \frac{||g||_\infty}{2\pi t}\int_{B(0,a)} \frac{1}{(1 - (1/2)^2)^{1/2}} dy \\ &\le \frac{||g||_\infty}{2\pi t}\frac{2}{\sqrt{3}}\pi a^2 = \frac{||g||_\infty a^2}{\sqrt{3}}\frac{1}{t} \end{aligned}$$

(b) Fix $x \in \mathbb{R}^2$ and consider $u(x,t)$ given by Poisson formula, then

$$tu(x,t) = \frac{1}{2\pi}\int_{B(x,t)} \frac{g(y)}{(1-(|y-x|/t)^2)^{1/2}}dy$$

If $t > 2(|x| + a)$, then we can consider the integral over $B(0,a)$ since $g(y) = 0$ outside of $B(0,a)$ and $B(0,a) \subseteq B(x,t)$. Moreover

$$\frac{g(y)}{(1-(|y-x|/t)^2)^{1/2}}$$

is bounded in $B(0,a)$. Therefore,

$$\begin{aligned}
\lim_{t\to\infty} tu(x,t) &= \lim_{t\to\infty} \frac{1}{2\pi}\int_{B(0,a)} \frac{g(y)}{(1-(|y-x|/t)^2)^{1/2}}dy \\
&= \frac{1}{2\pi}\int_{B(0,a)} \lim_{t\to\infty} \frac{g(y)}{(1-(|y-x|/t)^2)^{1/2}}dy \\
&= \frac{1}{2\pi}\int_{B(0,a)} g(y)dy = \frac{1}{2\pi}\int_{\mathbb{R}^2} g(y)dy
\end{aligned}$$

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