Take-Home Exam

Due: Wed Oct 30th by 6:00pm

Problem 1 (Continuous Dependence): Let $U \in \mathbb{R}^n$ be a bounded domain. Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad U \\ u = g & \text{on} \quad \partial U. \end{cases}$$

Assume f can be continuosly extended to ∂U (i.e. it is uniformly continuous).

(a) Show that the solution u has continuous dependence in $C(\overline{U})$ on the boundary data $g \in C(\partial U)$ and the source term $f \in C(\overline{U})$. Assume that a solution u always exists. Namely, show that if $g_n \to g$ in $C(\partial U)$ and $f_n \to f$ in $C(\overline{U})$ that the solution $u_n \to u$ in $C(\overline{U})$.

(b) Show that the solution has continuous dependence in $C^k(\overline{U})$ on the boundary data $g \in C^k(\partial U)$ and the source term $f \in C^k(\overline{U})$ for any $k \geq 1$ (again assume f and its first k derivatives can be continuously extended to $\overline{U}$).

Problem 2 (Greens Function): Let $U \subseteq \mathbb{R}^n$ have C^1 boundary (but is not necessarily bounded). Recall, the Neumann boundary value problem takes the form

$$\begin{cases} \Delta u = 0 & \text{in} \quad U \\ \frac{\partial u}{\partial \nu} = f & \text{on} \quad \partial U, \end{cases}$$

for some function $f \in C(\partial U)$ with $\int f \mathrm{d}S(y) = 0$, where ν denotes the outward facing normal to ∂U. Note that $\int_{\partial U} f \mathrm{d}S(y)$ is necessary since by the divergence theorem

$$\int_{\partial U} f(y)\mathrm{d}S(y) = \int_{\partial U} \partial_\nu u(y)\mathrm{d}S(y) = \int_U \Delta u \,\mathrm{d}y = 0$$

(a) (Extra Credit) We seek a Green's function $G(x,y)$ that can express a solution $u \in C^2(\overline{U})$ as

$$u(x) = c + \int_{\partial U} G(x,y)f(y)\mathrm{d}S(y),$$

where f is assumed to have compact support on ∂U, $\int_{\partial U} f \mathrm{d}S(y) = 0$ and c is a constant that depends on u. Give a formula for the Green's function in terms of a corrector $\phi^x(y)$ that is chosen to satisfy a particular BVP. (Hint, you may take for granted the formula we proved in class

$$u(x) = \int_{\partial U} \partial_\nu \Phi(x-y)u(y) - \partial_\nu u(y)\Phi(x-y)\mathrm{d}S(y),$$

where Φ is the fundamental solution.)

(b) Consider the Neumann problem in the upper half-plane $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$,

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^2_+ \\ u_{x_2} = f & \text{on} \quad \{x_2 = 0\}. \end{cases}$$

Find the corresponding Green's function and show that

$$u(x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}} \ln[(x_1 - y)^2 + x_2^2] \, f(y) \, \mathrm{d}y.$$

is a solution.

Problem 3 (Soap bubble). Let U be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and let φ be a smooth function on $\overline{U}$ satisfying $\varphi|_{\partial U} < 0$. Define the constraint set

$$\mathcal{A}_\varphi := \{u \in C^2(\overline{U}) \, : \, u|_{\partial U} = 0, \, u \geq \varphi \, \text{on} \, \overline{U}\}$$

and the surface energy of u

$$I[u] := \frac{1}{2} \int_U |Du(x)|^2 \mathrm{d}x.$$

We seek to minimize the surface energy $I[u]$ over all functions $u \in \mathcal{A}_\varphi$ constrained to lie above φ. Physically, we can view the minimizer as the shape of a "soap bubble" attached to the boundary ∂U and stretched over the obstacle φ. The soap bubble will try to minimize its energy as best it can subject to constraint imposed by the "obstacle".

Show that if such a minimizer exists, i.e. if there is a $u_* \in \mathcal{A}_\varphi$ such that

$$I[u_*] = \min\{I[u] \, : \, u \in \mathcal{A}_\varphi\},$$

then u_* is harmonic on the open set $V_\varphi := \{x \in U \, : \, u^* > \varphi\}$, namely u_* is harmonic wherever it doesn't touch φ. (Hint: for each test function ϕ compactly supported in $\{u_* > \varphi\}$, show that you can find and interval $(-\delta, \delta)$ of 0, depending on ϕ such that $u_* + \tau\phi > \varphi$ for all $\tau \in (-\delta, \delta)$.)

Problem 4 (Regularization and decay): Let u be the solution of Cauchy problem for the heat equation on $\mathbb{R}^n$

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u = f & \text{on} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

where $f \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p \in [1, \infty]$.Show that for each $r \geq p$ and multi-index α, there exist a constant $C \geq 0$ (independent of u) such that

$$\|D^\alpha u(\cdot, t)\|_{L^r} \leq \frac{C}{t^{\frac{|\alpha|}{2} + \frac{n}{2}(\frac{1}{p} - \frac{1}{r})}} \|f\|_{L^p}$$

(Hint 1: You may want to use Young's convolution inequality $\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$), where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.)
(Hint 2: Let $\phi(x) = \exp(-|x|^2/4)$. You may find it useful to show by induction that for any multi-index α there exists a polynomial $P(x)$ (whose exact form isn't important)

$$D^\alpha(\phi(x/\sqrt{t})) = \frac{1}{t^{|\alpha|/2}} P(x/\sqrt{t}) \phi(x/\sqrt{t}).$$

You may also find it useful to show that for any polynomial $P(y)$, there is a constant C such that

$$|P(y)\phi(y)| \leq C\phi(y/2).$$

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Problem 5 (Wave energy): Use an energy method to show that there is at most one smooth solution to the equation

$$u_{tt} + cu_t - u_{xx} = f$$

in the domain $(0, 1) \times (0, \infty)$ with initial conditions $u(x, 0) = g(x)$, $u_t(x, 0) = h(x)$ and boundary conditions $u(0, t) = u(1, t) = 0$. (Hint: there are different energies for $c > 0$ and $c < 0$)

Problem 6 (Wave decay): Let u be a C^2 solution of $u_{tt} - \Delta u = 0$ in $\mathbb{R}^2 \times (0, \infty)$, with

$$u(x, 0) = 0 \quad \text{and} \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}^2$$

where g is a smooth function satisfying $g(x) = 0$ for $|x| > a$.

(a) Show that there exists a constant C such that $|u(x, t)| \leq \frac{C}{t}$ for $t \geq 2(|x| + a)$

(b) Show that $\lim_{t \to \infty} tu(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} g(y)dy$ for all $x \in \mathbb{R}^2$.