PDE Problem Set 1

Problem 1. Complete Problem 1 in Chapter 1 Evans p12, classifying the following equations in §1.2, a) Linear equations 1,3,6,7,9,12,14, b) Nonlinear Equations 1, 3, 4, 5, 7,8, 9, 12, 13.

Problem 2. Suppose that g is a C^1 function. Find and explicit formula for the solution of the initial value problem.

$$\begin{cases} u_t + \mathbf{b} \cdot Du + cu = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^n$ are constants and $g \in C^1(\mathbb{R}^n)$.

Solution. We treat this as a first order ODE in times and multiply the equation through by the integrating factor e^{-ct} to obtain

$$e^{-ct} u_t e^{-ct} cu + e^{-ct} \mathbf{b} \cdot Du = 0$$

which becomes

$$\partial_t (e^{-ct} u) + \mathbf{b} \cdot D(e^{-ct} u) = 0$$

by a reversal of the product rule. Denoting $v(x, t) = e^{-ct} u(x, t)$, we have the following linear transport equation IVP

$$\begin{cases} v_t + \mathbf{b} \cdot Dv = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ v = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

which can be solved by characteristics in the usual way to give

$$v = g(x - \mathbf{b}t).$$

Therefore the explicit solution to the original equation is

$$u = e^{-ct} g(x - \mathbf{b}t).$$

■

Problem 3. Suppose that $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1(\mathbb{R}^n)$ and let $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$ be the flow map that solves the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi^t(x) = \mathbf{b}(\phi^t(x)), \quad \phi^0(x) = x,$$

namely ϕ^t is the map that sends the initial point $x \in \mathbb{R}^n$ to the solution $x_t \in \mathbb{R}^n$ of the ODE $\dot{x} = \mathbf{b}(x)$ at time $t \in \mathbb{R}$.

a) Show that the initial value problem

$$\begin{cases} u_t + \mathbf{b} \cdot \nabla u = 0 & \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+ \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

has explicit solution

$$u = g \circ \phi^{-t}.$$

b) Show that the initial value problem

$$\begin{cases} u_t + \mathbf{b} \cdot \nabla u = f & \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+ \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

has explicit solution

$$u = g \circ \phi^{-t} + \int_0^t f \circ \phi^{s-t} \mathrm{d}s.$$

Solution:

a) This was shown in class. It follows since $\mathrm{d}u(t, \phi^t(x))/\mathrm{d}t = 0$ and therefore $u(t, \phi^t(x)) = g(x)$. Since $(\phi^t)^{-1} = \phi^{-t}$ we obtain $u(t, x)g(\phi^t(x))$. ∎

b) We follow a similar strategy to part (a). We have

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t, \phi^t(x)) = f(\phi^t(x)).$$

Integrating both sides from 0 to t gives

$$u(t, \phi^t(x)) - u(0, x) = \int_0^t f(\phi^s(x))\mathrm{d}s.$$

Composing with the inverse ϕ^{-t} and using that $\phi^s \circ \phi^{-t} = \phi^{s-t}$ gives the result.

Problem 4. Recall that the integration by parts formula in appendix C of Evans

$$\int_V uv_{x_i}\,\mathrm{d}x = \int_{\partial V} uv\nu^i\,\mathrm{d}S - \int_V u_{x_i}v\,\mathrm{d}x$$

holds for all *bounded* domains $V \subseteq \mathbb{R}^n$ and C^1 functions u, v. However in lecture (and in Evan's proof in Chapter 2) when we proved that

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,\mathrm{d}y$$

solved Poisson's equation $-\Delta u = f$, by applying the integration parts formula on the *unbounded domain* $\mathbb{R}^n \backslash B(0,\epsilon)$. Rewrite the proof of Theorem 1 part (ii) in Chapter 2 correctly so that you only apply integration by parts on a bounded domain. //

Solution: The proof will follow along the same lines as in the text and the singularity at 0 will be dealt with the same way. To get a bounded domain, we will also truncate to a ball $B(x,R)$. We will use the integration by parts formula on the ball $B(x,R)$ for $R > 0$ and then take the limit as $R \to \infty$. In this way we will avoid the singularity at 0 and the unbounded domain. Following this strategy, we generate three additional terms

$$\int_{B(0,R)^c} \Phi(x-y)\Delta f(y)\,\mathrm{d}y, \quad \int_{\partial B(0,R)} \Phi(x-y)\frac{\partial f}{\partial \nu}\,\mathrm{d}S(y), \quad \int_{\partial B(0,R)} \frac{\partial \Phi}{\partial \nu}(x-y)f(y)\,\mathrm{d}S(y).$$

Since f is assumed to have compact support, the first second and third terms all vanish identically as $R \to \infty$. ■

Problem 5. Prove that the Laplace equation $\Delta u = 0$ is rotation invariant. That is if $\mathbf{R}$ is an orthogonal $n \times n$ matrix, and we define

$$v(x) = u(\mathbf{R}x), \quad x \in \mathbb{R}^n,$$

then $\Delta v = 0$.

Solution: We will prove this by direct computation. Note orthogonality of $\mathbf{R}$ means that $\mathbf{R}\mathbf{R}^T = I$ (or $\sum_{i=1}^n R_{j,i}R_{k,i} = \delta_{jk}$). By the chain rule, we have

$$\frac{\partial v}{\partial x_i} = \sum_{j=1}^n \frac{\partial u}{\partial y_j} R_{ji}$$

and

$$\frac{\partial^2 v}{\partial x_i^2} = \sum_{j,k=1}^n \frac{\partial^2 u}{\partial y_j y_k}(\mathbf{R}x)R_{ji}R_{ki}.$$

Therefore

$$\sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2} = \sum_{i,j,k=1}^n \frac{\partial^2 u}{\partial y_j \partial y_k}(\mathbf{R}x)R_{ji}R_{ki} = \sum_{j,k=1}^n \frac{\partial^2 u}{\partial y_j \partial y_k}(\mathbf{R}x)\delta_{jk} = \Delta u(\mathbf{R}x).$$

Problem 6. We say that a function $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in} \quad U.$$

(a) Prove that if v is subharmonic then

$$v(x_0) \leq \fint_{B(x_0,r)} v(x)dx \quad \text{for all} \quad r > 0.$$

(b) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth and convex function. Assume that u is harmonic and set $v = \phi(u)$. Prove that v is subharmonic.

Solution:

(a) Following the proof of mean value property for harmonic functions, we have define

$$\phi(r) = \fint_{\partial B(x_0,r)} v(x)\mathrm{d}S(x),$$

taking the derivative and using the divergence theorem gives

$$\phi'(r) = \fint_{B(x_0,r)} \Delta v \mathrm{d}x \geq 0.$$

Therefore $\phi(r)$ is increasing in r and the result follows and so

$$u(x_0) = \lim_{r\to 0} \phi(r) \leq \phi(r) = \fint_{\partial B(x_0,r)} v(x)\mathrm{d}S(x)$$

The result follows by integrating both sides from 0 to r as in the proof of the harmonic case.

(b) Since ϕ is convex, we have that $\phi''(u) \geq 0$. Therefore

$$\Delta v = \Delta(\phi(u)) = \phi'(u)\Delta u + \phi''(u)|Du|^2 \geq 0.$$