PDE Problem Set 1

Due: Fri Sept 06

Problem 1. Complete Problem 1 in Chapter 1 Evans p12, classifying the following equations in §1.2, a) Linear equations 1,3,6,7,9,12,14, b) Nonlinear Equations 1, 3, 4, 5, 7,8, 9, 12, 13.

Problem 2. Suppose that g is a C^1 function. Find and explicit formula for the solution of the initial value problem.

$$\begin{cases} u_t + \mathbf{b} \cdot Du + cu = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^n$ are constants and $g \in C^1(\mathbb{R}^n)$.

Problem 3. Suppose that $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1(\mathbb{R}^n)$ and let $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$ be the flow map that solves the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi^t(x) = \mathbf{b}(\phi^t(x))\, \phi^0(x) = x,$$

namely ϕ^t is the map that sends the initial point $x \in \mathbb{R}^n$ to the solution $x_t \in \mathbb{R}^n$ of the ODE $\dot{x} = \mathbf{b}(x)$ at time $t \in \mathbb{R}$.

a) Show that the initial value problem

$$\begin{cases} u_t + \mathbf{b} \cdot \nabla u = 0 & \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+ \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

has explicit solution

$$u = g \circ \phi^{-t}.$$

b) Show that the initial value problem

$$\begin{cases} u_t + \mathbf{b} \cdot \nabla u = f & \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+ \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

has explicit solution

$$u = g \circ \phi^{-t} + \int_0^t f \circ \phi^{s-t} \mathrm{d}s.$$

Problem 4. Recall that the integration by parts formula in appendix C of Evans

$$\int_V uv_{x_i}\,\mathrm{d}x = \int_{\partial V} uv\nu^i\,\mathrm{d}S - \int_V u_{x_i}v\,\mathrm{d}x$$

holds for all *bounded* domains $V \subseteq \mathbb{R}^n$ and C^1 functions u, v. However in lecture (and in Evan's proof in Chapter 2) when we proved that

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,\mathrm{d}y$$

solved Poisson's equation $-\Delta u = f$, by applying the integration parts formula on the *unbounded domain* $\mathbb{R}^n \backslash B(0,\epsilon)$. Rewrite the proof of Theorem 1 part (ii) in Chapter 2 correctly so that you only apply integration by parts on a bouned domain.

Problem 5. Prove that the Laplace equation $\Delta u = 0$ is rotation invaraiant. That is if $\mathbf{R}$ is an orthogonal $n \times n$ matrix, and we define

$$v(x) = u(\mathbf{R}x), \quad x \in \mathbb{R}^n,$$

then $\Delta v = 0$.

Problem 6. We say that a function $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in} \quad U.$$

(a) Prove that if v is subharmonic then

$$v(x_0) \leq ⨍_{B(x_0,r)} v(x)dx \quad \text{for all} \quad r > 0.$$

(b) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth and convex function. Assume that u is harmonic and set $v = \phi(u)$. Prove that v is subharmonic.