PDE Problem Set 2 - Solutions

Problem 1: Show that if v is subharmonic $-\Delta v \leq 0$, then v still satisfies the weak maximum principle

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

(Hint: Recall problem 6 from last homework). Show by explicit example that a subharmonic function need not satisfy the weak minimum principle.

Solution: From the last homework, we know that subharmonic functions have the following sub mean value property

$$v(x) \leq \fint_{B(x,r)} v \, \mathrm{d}y$$

for any ball $B(x,r) \subset U$. The proof of the maximum principle now follows exactly along the same lines as the proof in the harmonic case. We repeat it here for convenience. We first assume that U is connected by working on each connected component separately. Recall, if we assume v achieves its max $M = \max_{\overline{U}} u = u(x^0)$ at a point $x^0 \in U$, then the closed set $V = \{u = M\}$ is non-empty. Fix an $x \in V$, by the sub mean value formula we have

$$M = v(x) \leq \fint_{B(x,r)} v \, \mathrm{d}y \leq M$$

for an appropriate choice of r with equality in the case that $v = M$ on $B(x,r)$. It follows that $B(x,r) \subset V$ and therefore V is also an open set. Since U was assumed connected it follows that $U = V$ and therefore u is constant on U. The weak maximum principle applies by applying this result on each connected component.

To see that a sub harmonic function need not satisfy the minimum principle, we note that the function $u = |x|^2$ on $B(0,1)$ satisfies

$$-\Delta u = -2n \leq 0$$

but it has a mimimum value 0 at $x = 0$ but takes the value 1 on the boundary. ∎

Problem 2: Let U be a bounded open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\overline{U}} |u| \leq C(\max_{\partial U} |g| + \max_{\overline{U}} |f|)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

(Hint: Show that $v = u + \frac{|x|^2}{2n}\lambda$ is subharmonic for $\lambda = \max_{\bar{U}} |f|$).

Solution: According to the hint, and the fact that $\Delta|x|^2 = 2n$ we immediately see that $v = u + \frac{|x|^2}{2n}\lambda$, $\lambda = \max_{\overline{U}} |f|$ satisfies

$$-\Delta v = f - \max_{\overline{U}} |f| \leq 0$$

and is therefore subharmonic and satisfies the maximum principle $\max_{\overline{U}} v = \max_{\partial U} v$. This implies that

$$\max_{\overline{U}} u \leq \max_{\overline{U}} v = \max_{\partial U} \left(g + \frac{|x|^2}{2n}\lambda \right) \leq \max_{\partial U} g + \frac{\max_{\partial U} |x|^2}{2n}\lambda.$$

Similarly if we instead define $v = u - \frac{1}{2n}|x|^2\lambda$, then

$$-\Delta v = f + \max_{\overline{U}} |f| \geq 0,$$

and so v superharmonic and satisfies the minimum principle (or alternatively $-v$ satisfies the maximum principle) $\min_{\overline{U}} v = \min_{\partial U} v$. This implies that

$$\min_{\overline{U}} u \geq \min_{\overline{U}} v = \min_{\partial U} \left(g - \frac{|x|^2}{2n}\lambda \right) \geq \min_{\partial U} g - \frac{\max_{\partial U} |x|^2}{2n}\lambda.$$

Denoting $C = \max\{1, \frac{\max_{\partial U} |x|^2}{2n}\}$ we find

$$\max_{\overline{U}} u \leq C(\max_{\partial U} g + \max_{\overline{U}} |f|)$$

as well as

$$-\min_{\overline{U}} u \leq C(-\min_{\partial U} g + \max_{\overline{U}} |f|).$$

Combining these gives the result. ∎

Problem 3. The Kelvin transform $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u}(x) := u(\bar{x})|\bar{x}|^{n-2} = u(x/|x|^2)|x|^{2-n}, \quad x \neq 0,$$

where $\bar{x} = x/|x|^2$ is the inversion through the unit sphere. Show that if u is harmonic, then so is $\bar{u}$. (Hint: First show that $D_x\bar{x}(D_x\bar{x})^\top = |\bar{x}|^4 I$, namely the mapping $x \mapsto \bar{x}$ is conformal, meaning it preserves angles.)

Solution: First we show that $D_x\bar{x}(D_x\bar{x})^\top = |\bar{x}|^4 I$. It is helpful to see this in coordinate notation, first we see that

$$\partial_{x_j}\bar{x}_i = \frac{\delta_{ij}}{|x|^2} - \frac{2x_i x_j}{|x|^4},$$

where δ_{ij} is 1 if $i = j$ and is 0 if $i \neq j$ (known as the Kronecker delta). In coordinate free notation, we can write this as

$$D_x\bar{x} = |x|^{-2}\left(I - \frac{2xx^\top}{|x|^2}\right) = (|\bar{x}|^2 I - 2\bar{x}\bar{x}^\top).$$

Now we compute

$$D_x\bar{x}(D_x\bar{x})^\top = |\bar{x}|^2(I - 4\bar{x}\bar{x}^\top + 4\bar{x}\bar{x}^\top) = |\bar{x}|^4 I.$$

This means that the transformation $x \mapsto \bar{x}$ is a conformal map (meaning it locally preserves angles).

Lets see what happens to the Laplacian when we change coordinates under this conformal map $x \mapsto \bar{x}$. Given a function $f(\bar{x})$, we readily find

$$\Delta_x f = \sum_{\ell,m=1}^{n} (D_x\bar{x}_\ell \cdot D_x\bar{x}_m)\partial_{\bar{x}_m}\partial_{\bar{x}_\ell} f + D_{\bar{x}} f \cdot \Delta_x \bar{x}.$$

Using the conformal property $D_x\bar{x}(D_x\bar{x})^\top = |\bar{x}|^4 I$ gives

$$\Delta_x f = |\bar{x}|^4 \Delta_{\bar{x}} f + \Delta_x\bar{x} \cdot D_{\bar{x}} f \tag{1}$$

Next we compute $\Delta_x\bar{x}$, we find

$$\begin{aligned}
\Delta_x\bar{x}_i &= \sum_{j=1}^{n} \partial_{x_j}(|x|^{-2}\delta_{ij} - 2x_i x_j|x|^{-4}) \\
&= -2x_i|x|^{-4} - 2x_i|x|^{-4} - 2nx_i|x|^{-4} + 8x_i|x|^{-4} \\
&= 2(2-n)x_i|x|^{-4} \\
&= 2(2-n)\bar{x}_i|\bar{x}|^2
\end{aligned}$$

Substituting this into (1) gives

$$\Delta_x f = |\bar{x}|^4 \Delta_{\bar{x}} f + 2(2-n)|\bar{x}|^2 \bar{x} \cdot D_{\bar{x}} f.$$

Note that when $n = 2$, this immediately gives that $\bar{f}(x) = f(\bar{x})$ is harmonic whenever f is. When $n \geq 3$, we instead have an extra term. To account for this, we note that $D_{\bar{x}}|\bar{x}|^{2-n} = (2-n)\bar{x}|\bar{x}|^{-n}$ as well as $\Delta_{\bar{x}}|\bar{x}|^{2-n} = 0$ away from $\bar{x} = 0$ (since $|x|^{2-n}$ is the fundamental solution). This means that we can write

$$\Delta_x f = |\bar{x}|^{2+n}\big(|\bar{x}|^{2-n}\Delta_{\bar{x}} f + 2D_{\bar{x}}|\bar{x}|^{2-n} \cdot D_{\bar{x}} f + \underbrace{\Delta_{\bar{x}}|\bar{x}|^{2-n}}_{=0} f\big).$$

Using that $\Delta(fg) = \Delta f g + 2Df \cdot Dg + f\Delta g$ gives the final identity

$$\Delta_x f = |\bar{x}|^{2+n}\Delta_{\bar{x}}(|\bar{x}|^{2-n} f), \quad \bar{x} \neq 0.$$

This implies the result to be proved.

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Problem 4: Use Poisson's formula for the ball to prove that if u is positive and harmonic in the open ball $B(0,r)$, then

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0).$$

This is an explicit form of Harnack's inequlity.

Solution: Since u is harmonic in the ball $B(0,r)$, then by Poisson's formula for the ball $u(x)$ is given by

$$\begin{aligned} u(x) &= \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}dS(y) \\ &= r^{n-2}(r^2-|x|^2)⨍_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}dS(y) \end{aligned}$$

for $x \in B^0(0,r)$. Since for $y \in B(0,r)$ and $x \in B^0(0,r)$, we have by the triangle and reverse triangle inequalities

$$\frac{1}{(r+|x|)^n} \leq \frac{1}{|x-y|^n} \leq \frac{1}{(r-|x|)^n}$$

Therefore using the fact that u is always positive, we can bound $u(x)$ above in $B(0,r)$ by

$$\begin{aligned} u(x) &\leq r^{n-2}\frac{(r-|x|)(r+|x|)}{(r-|x|)^n}⨍_{\partial B(0,r)} u(y)dS(y) \\ &= r^{n-2}\frac{(r+|x|)}{(r-|x|)^{n-1}}u(0) \end{aligned}$$

where we have used Poisson formula on the ball at the point $x=0$ in the last step. Similarly we can bound $u(x)$ below in $B(0,r)$ by

$$\begin{aligned} u(x) &\geq r^{n-2}\frac{(r-|x|)(r+|x|)}{(r+|x|)^n}⨍_{\partial B(0,r)} u(y)dS(y) \\ &= r^{n-2}\frac{(r-|x|)}{(r+|x|)^{n-1}}u(0). \end{aligned}$$

Both of these bounds constitute an explicit form of Harnack's inequality.

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Problem 5. Let U^+ denote the open half ball $U^+ = \{x \in \mathbb{R}^n : |x| < 1 \text{ and } x_n > 0\}$. Assume that $u \in C^2(U^+) \cap C(\overline{U^+})$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } U^+ \\ u = 0 & \text{on } \partial U^+ \cap \{x \in \mathbb{R}^n : x_n = 0\}. \end{cases}$$

Extend u to the ball $U = B(0,1)$ by reflecting across the $x_n = 0$ plane via

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

Prove that $v \in C^2(U)$ and that v is harmonic in U. (Hint: use Poisson's formula for the ball to obtain a candidate harmonic function w and then apply the maximum principle on each half of the ball to show that $w = v$)

Solution: By Poisson's formula for the ball we construct a Harmonic function on $B(0,1)$ via

$$w(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{v(y)}{|x - y|^n} \mathrm{d}S(y).$$

Note that w inherits the antisymmetry of v, $w(\tilde{x}) = -w(x)$, $\tilde{x} = (x_1, \ldots, -x_n)$ since $x \mapsto \tilde{x}$ preserves the ball boundary $\partial B(0,1)$ and $|\tilde{x}| = |x|$, therefore by changing variables in the integral

$$w(\tilde{x}) = \frac{1 - |\tilde{x}|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{v(y)}{|\tilde{x} - y|^n} \mathrm{d}S(y) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{v(\tilde{y})}{|\tilde{x} - \tilde{y}|^n} \mathrm{d}S(y) = -w(x).$$

This anti-symmetry implies that $w = 0$ on $\partial U^+ \cap \{x_n = 0\}$. It follows that the function $h = v - w$ is harmonic on U^+ and U^- separately and satisfies $h = 0$ on ∂U^+ and ∂U^-. By the maximum principle (or uniqueness) we see that $h = 0$ on $\overline{U^+}$ and $\overline{U^-}$ and therefore $w = v$ on $\overline{B(0,1)} = \overline{U^+} \cup \overline{U^-}$, implying that v is C^2 and harmonic. ∎