PDE Problem Set 2

Due: Fri Sept 20

Problem 1: Show that if v is subharmonic $-\Delta v \leq 0$, then v still satisfies the weak maximum principle

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

(Hint: Recall problem 6 from last homework). Show by explicit example that a subharmonic function need not satisfy the weak minimum principle.

Problem 2: Let U be a bounded open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\overline{U}} |u| \leq C(\max_{\partial U} |g| + \max_{\overline{U}} |f|)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

(Hint: Show that $v = u + \frac{|x|^2}{2n}\lambda$ is subharmonic for $\lambda = \max_{\overline{U}} |f|$).

Problem 3. The Kelvin transform $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u}(x) := u(\bar{x})|\bar{x}|^{n-2} = u(x/|x|^2)|x|^{2-n}, \quad x \neq 0,$$

where $\bar{x} = x/|x|^2$ is the inversion through the unit sphere. Show that if u is harmonic, then so is $\bar{u}$. (Hint: First show that $D_x\bar{x}(D_x\bar{x})^\top = |\bar{x}|^4 I$, namely the mapping $x \mapsto \bar{x}$ is conformal, meaning it preserves angles.)

Problem 4: Use Poisson's formula for the ball to prove that if u is positive and harmonic in the open ball $B(0, r)$, then

$$r^{n-2}\frac{r - |x|}{(r + |x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r + |x|}{(r - |x|)^{n-1}}u(0).$$

This is an explicit form of Harnack's inequlity.

Problem 5. Let U^+ denote the open half ball $U^+ = \{x \in \mathbb{R}^n : |x| < 1 \text{ and } x_n > 0\}$. Assume that $u \in C^2(U^+) \cap C(\overline{U^+})$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } U^+ \\ u = 0 & \text{on } \partial U^+ \cap \{x \in \mathbb{R}^n : x_n = 0\}. \end{cases}$$

Extend u to the ball $U = B(0,1)$ by reflecting across the $x_n = 0$ plane via

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

Prove that $v \in C^2(U)$ and that v is harmonic in U. (Hint: use Poisson's formula for the ball to obtain a candidate harmonic function w and then apply the maximum principle on each half of the ball to show that $w = v$)