PDE Problem Set 3

Due: Wed Oct 2

Problem 1: Let U be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary and let $f \in C^0(U)$. Consider the energy functional

$$I[u] = \int_U \sqrt{1 + |Du|^2} - f\, u\, \mathrm{d}x$$

and set $\mathcal{A} = \{u \in C^2(\overline{U}) | u = g \text{ on } \partial U\}$. Show that if $u \in \mathcal{A}$ is such that

$$I[u] = \min_{w \in \mathcal{A}} I[w]$$

then u solves the following equation

$$-\text{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = f \quad \text{in} \quad U.$$

Problem 2: Let U be a connected bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Consider the boundary value problem

$$\begin{cases} -\Delta u + K^2 u = 0 & \text{in } \mathcal{U} \\ u + Du \cdot \nu = \phi & \text{on } \partial \mathcal{U} \end{cases}$$

where $\phi \in C(\partial U)$ and $K \in \mathbb{R}$ is a scalar.

(a) Use an energy method to show that there exists at most one solution in $C^2(\overline{U})$ of this boundary value problem.

(b) Show that if $u \in C^2(\overline{U})$ satisfies the nonhomogeneous problem

$$\begin{cases} -\Delta u + K^2 u = f & \text{in } U \\ u + Du \cdot \nu = 0 & \text{on } \partial \mathcal{U} \end{cases}$$

then u minimizes the functional

$$I[w] = \int_U \frac{1}{2}|Dw|^2 + \frac{1}{2}K^2 w^2 - f w\, dx + \int_{\partial \mathcal{U}} \frac{1}{2} w^2 dS$$

on the set $\mathcal{A} = \{w \in C^2(U) | w + Dw \cdot \nu = 0 \text{ on } \partial U\}$.

Problem 3: Assume U is an open, bounded, connected subset of $\mathbb{R}^n$ with C^1 boundary. Use an energy method to show that if u is a smooth solution of the Neumann boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in} \quad U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial U, \end{cases}$$

then u is constant in U.

Problem 4. We say v is a subsolution of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in} U_T.$$

(a) Prove that for a subsolution

$$v(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{|t-s|^2} \mathrm{d}y \mathrm{d}s$$

for all $E(x,t;r) \subset U_T$. (You may reference specific calculations already in the text)

(b) Prove that for a subsolution, the maximum principle still holds

$$\max_{\overline{U}_T} v = \max_{\Gamma_T} v.$$

Problem 5. Let $u \in C_1^2(U \times (0,\infty)) \cap C(\overline{U} \times [0,\infty))$ be a solution of

$$\begin{cases} u_t - \Delta u = \sin(u) & \text{in} \quad U \times (0,\infty) \\ u = 0 & \text{on} \quad \partial U \times (0,\infty) \\ u = g & \text{on} \quad U \times t = 0 \end{cases}$$

Show that if $g(x) \leq 1$, then $u(x,t) \leq e^t$ for all $x \in U$ and $t > 0$. (Hint: Show that $v(x,t) = u(x,t) - e^t$ is a subsolution of the heat equation and use the maximum principle you proved in problem 4b.)