PDE Problem Set 4

Due: Fri Dec 6th

Problem 1:

(a) Let $H(p) = \frac{1}{r}|p|^r$ for $1 < r < \infty$. Compute the Legendre transform of H^* of H.

(b) Let $H(p) = \frac{1}{2}\sum_{i,j=1}^n a_{ij}p_ip_j + \sum_{i=1}^n b_jp_i$, where $A = (a_{ij})$ is a symmetric positive definite matrix and $b \in \mathbb{R}^n$. Compute H^*.

Problem 2: Let H be convex. We say that v belongs to the *subdifferential* of H at p if

$$H(q) \geq H(p) + v \cdot (q - p) \quad \forall q \in \mathbb{R}^n,$$

we write this as $v \in \partial H(p)$. Prove that

$$v \in \partial H(p) \Leftrightarrow p \in \partial L(v) \Leftrightarrow p \cdot v = H(p) + L(v),$$

where $L = H^*$ is the Legendre transform.

Problem 3: Let u^1, u^2 be two solutions (given by the Hopf-Lax formula) of

$$u_t + H(Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

with initial condition $u^i = g^i$. Prove that

$$\sup_{x \in \mathbb{R}^n} |u^1(x,t) - u^2(x,t)| \leq \sup_{x \in \mathbb{R}^n} |g^1(x) - g^2(x)|, \quad \forall t \geq 0.$$

Problem 4: Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to this initial value problem

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = \begin{cases} 0 & \text{if } \; x \in E \\ +\infty & \text{if } \; x \notin E \end{cases} & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

It would give the solution

$$u(x,t) = \frac{1}{4t}\mathrm{dist}(x,E)^2.$$

Problem 5: Consider the initial value problem

$$\begin{cases} u_t + uu_x + u = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = a \sin x & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

(a) Find the characteristic curves explicitly.

(b) Show that if $a > 1$, then there does not exist a smooth solution defined for all time $t > 0$. Find the maximal time of existence of the smooth solution.

Problem 6: Find an integral solution of

$$\begin{cases} u_t + (F(u))_x = 0, & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

when $F(u) = u^2 + u$ and $g(x) = \begin{cases} 2 & x < 0 \\ -3 & x > 0 \end{cases}$.

Problem 7: Find an entropy solution of

$$\begin{cases} u_t + u^2 u_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

with $g(x) = \begin{cases} 0 & x < 0 \\ -2 & x > 0 \end{cases}$. Make sure your solution satisfies the entropy conditions.