PDE Problem Set 1

Due: Mon Feb 10

Problem 1: Completeness of Hölder Spaces

(a) Let $0 < \alpha \leq 1$. Show that the Hölder space $C^\alpha(\overline{\Omega})$ is a Banach space, where Ω is a bounded open subset of $\mathbb{R}^n$. Recall that $C^\alpha(\overline{\Omega})$ consists of continuous functions $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\|u\|_{C^\alpha(\overline{\Omega})} = \|u\|_{C(\overline{\Omega})} + \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty.$$

(b) Consider the space $C^{k,\alpha}(\overline{\Omega})$, where k is a positive integer and $0 < \alpha \leq 1$. This space consists of functions u whose partial derivatives up to order k exist and are continuous, and the k-th order partial derivatives are Hölder continuous with exponent α. Define a suitable norm on $C^{k,\alpha}(\overline{\Omega})$ and prove that it is a Banach space.

(c) Is the inclusion map $C^\beta(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega})$ continuous when $0 < \alpha < \beta \leq 1$? Is it compact?

Hint: For part (a), you may want to start by showing that if (u_n) is a Cauchy sequence in $C^\alpha(\overline{\Omega})$, then it is also Cauchy in $C(\overline{\Omega})$. Use the completeness of $C(\overline{\Omega})$ to find a candidate limit function u. Then show that u is indeed in $C^\alpha(\overline{\Omega})$ and that u_n converges to u in the Hölder norm. For part (c), consider using the Arzela-Ascoli Theorem.

Problem 2: Arzela-Ascoli Theorem on $\mathbb{R}^n$
Let $\mathcal{F}$ be a family of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that:

(i) (Pointwise Boundedness) For each $x \in \mathbb{R}^n$, the set $\{f(x) : f \in \mathcal{F}\}$ is bounded.

(ii) (Equicontinuity) For every $\epsilon > 0$ and every $x_0 \in \mathbb{R}^n$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all $f \in \mathcal{F}$ and all $x \in \mathbb{R}^n$ with $|x - x_0| < \delta$.

(iii) (Uniform decay at infinity) For every $\epsilon > 0$ there exists $R > 0$ such that for all $f \in \mathcal{F}$, $|f(x)| < \epsilon$ whenever $|x| > R$.

Prove that $\mathcal{F}$ is precompact in the space of continuous functions on $\mathbb{R}^n$ with the uniform norm.

Hint: You can adapt the proof of the Arzela-Ascoli Theorem for functions on a compact set to this case. Consider a sequence (f_n) in $\mathcal{F}$. First, apply the standard Arzela-Ascoli Theorem to the restrictions of the functions to the closed ball $\overline{B}(0, R)$ for increasing values of R. Use a diagonalization argument to extract a subsequence that converges uniformly on every closed ball. Finally, use the uniform decay condition (iii) to show that this subsequence converges uniformly on all of $\mathbb{R}^n$.

Problem 3: Bounded Linear Operators and Dual Spaces

(a) Let X and Y be Banach spaces, and let $T : X \to Y$ be a bounded linear operator. The adjoint operator $T^* : Y^* \to X^*$ is defined by $\langle T^*y^*, x\rangle = \langle y^*, Tx\rangle$ for all $y^* \in Y^*$ and $x \in X$. Prove that $\|T^*\| \le \|T\|$.

(b) Let $1 < p < \infty$ and let q be its conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual space of $L^p(\Omega)$ is isometrically isomorphic to $L^q(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^n$.

Hint: For part (a), use the definition of the adjoint operator. For part (b) you can use the Riesz Representation theorem for L^p spaces.

Problem 4: Uniform Convexity and Weak Convergence

A Banach space X is said to be uniformly convex if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$, if $\|x - y\| \ge \epsilon$, then $\left\|\frac{x+y}{2}\right\| \le 1 - \delta$.

(a) Prove that every uniformly convex Banach space is strictly convex. That is, if $x \ne y$ and $\|x\| = \|y\| = 1$, then for all $0 < t < 1$, $\|tx + (1-t)y\| < 1$.

(b) Show that $L^p(\Omega)$ is uniformly convex for $1 < p < \infty$, where Ω is a bounded open subset of $\mathbb{R}^n$.

(c) Let X be a uniformly convex Banach space. Suppose (x_n) is a sequence in X that converges weakly to $x \in X$, and $\lim_{n\to\infty} \|x_n\| = \|x\|$. Show that (x_n) converges strongly to x.

Hint: For part (a), consider the contrapositive. For part (b), you can use Clarkson's inequalities (which you may assume without proof and are written below). For part (c), consider the sequence $(\frac{x_n}{\|x_n\|})$ if $x \ne 0$.

Bonus: Prove Clarkson's Inequalities: Let $2 \le p < \infty$. Then for all $u, v \in L^p(\Omega)$,

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \le \frac{1}{2}\left(\|u\|_p^p + \|v\|_p^p\right).$$

Let $1 < p \leq 2$. Then for all $u, v \in L^p(\Omega)$,

$$\left\| \frac{u+v}{2} \right\|_p^q + \left\| \frac{u-v}{2} \right\|_p^q \leq \left(\frac{1}{2} \left(\|u\|_p^p + \|v\|_p^p \right) \right)^{q-1},$$

where q is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.
Hint: For the case $2 \leq p < \infty$, you can start by proving the inequality

$$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \leq \frac{1}{2} \left(|a|^p + |b|^p \right)$$

for all real numbers a and b. This can be done using calculus (consider the function $f(t) = |1+t|^p + |1-t|^p$ and show that it is maximized at $t = 0$ on the interval $[0, 1]$). Then, integrate the pointwise inequality

$$\left| \frac{u(x)+v(x)}{2} \right|^p + \left| \frac{u(x)-v(x)}{2} \right|^p \leq \frac{1}{2} \left(|u(x)|^p + |v(x)|^p \right)$$

over Ω.

For the case $1 < p \leq 2$, you can use duality. First, prove the inequality for $p = 2$. Then, use the fact that the dual space of L^p is L^q and the result for $2 \leq q < \infty$ to deduce the inequality for $1 < p < 2$.

Problem 5: Riesz-Kolmogorov Compactness Criterion in $L^1(\mathbb{R})$
Let $\Omega = (0, 1)$ and consider the sequence of functions (f_n) in $L^1(\mathbb{R})$ defined by

$$f_n(x) = \begin{cases} n & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{otherwise} \end{cases}$$

for $n = 1, 2, 3, \ldots$.

(a) Show that the sequence (f_n) is bounded in $L^1(\mathbb{R})$.

(b) Show that the sequence (f_n) is not relatively compact in $L^1(\mathbb{R})$ by directly considering the definition of relative compactness (i.e., by showing that there is no convergent subsequence).

(c) Determine which of the conditions of the Riesz-Kolmogorov compactness criterion fails for the sequence (f_n). Provide a detailed justification for your answer.