Ordinary Differential Equations

1 Introduction: first order ODE

We are given

- a function $f(t, y)$ which describes a "direction field" in the (t, y) plane
- an initial point (t_0, y_0)

We want to find a function $y(t)$ for $t \in [t_0, T]$ such that

- $y(t_0) = y_0$ "initial condition"
- $y'(t) = f(t, y(t))$ for $t \in [t_0, T]$ "ordinary differential equation" (ODE)

This is called an **initial value problem** (IVP).

The partial derivative $f_y(t, y) = \dfrac{\partial f}{\partial y}(t, y)$ is important for the behavior of the differential equation.

Theorem 1.1. *Assume that $f(t, y)$ and $f_y(t, y)$ are continuous for $t \in [t_0, T]$, $y \in \mathbb{R}$.*
For a given initial value y_0 there is a unique solution $y(t)$ of the initial value problem. Either the solution exists for all $t \in [0, T]$, or it only exists on a smaller interval $[t_0, t_*)$ with $t_0 < t_* < T$.

We can solve the IVP in Matlab with **ode45**:

```
f = @(t,y) ...                    % define function f(t,y)
[ts,ys] = ode45(f,[t0,T],y0);     % find column vectors ts,ys with values of solution
ys(end)                           % value y(T) of solution
plot(ts,ys)                       % plot solution
```

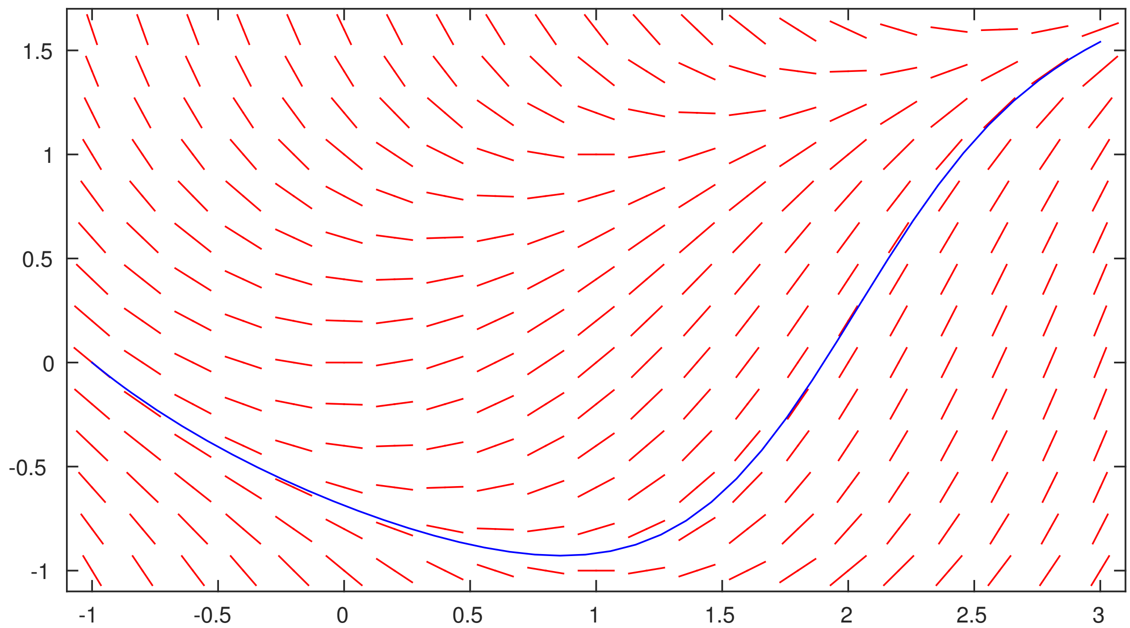
Example: Find a function $y(t)$ for $t \in [-1, 3]$ such that

$$\begin{aligned} y'(t) &= t - y(t)^2 \\ y(-1) &= 0 \end{aligned}$$

Here we have $t_0 = -1$, $y_0 = 0$, $f(t, y) = t - y^2$.

Numerical solution in Matlab: (using m-file `dirfield.m` from course web page)

```
f = @(t,y) t-y^2                          % define function f(t,y)
dirfield(f, -1:.2:3, -1:.2:1.6); hold on  % plot direction field
[ts,ys] = ode45(f,[-1,3],0);              % solve IVP for t from -1 to 3, initial value 0
                                          % this gives vectors ts,ys
plot(ts,ys,'b'); hold off                 % plot solution
```



What happens if we perturb the initial value y_0?

Theorem 1.2. *Let $y(t)$ denote the solution of the IVP with initial condition $y(t_0) = y_0$, let $\tilde{y}(t)$ denote the solution of the IVP with initial condition $\tilde{y}(t_0) = \tilde{y}_0$. Assume* $\boxed{f_y(t, y) \le M}$ *for $t \in [t_0, T]$, $y \in \mathbb{R}$. Then*

$$|\tilde{y}(t) - y(t)| \le |\tilde{y}_0 - y_0| \, e^{M(t-t_0)} \qquad \text{for } t \in [t_0, T]$$

For $M < 0$ the difference $|\tilde{y}(t) - y(t)|$ decays exponentially for increasing t. For $M > 0$ the difference may increase exponentially.

We call the ODE **unstable** if we have $\boxed{f_y(t, y) > 0}$ for all $t \in [t_0, T]$, $y \in \mathbb{R}$.

We call the ODE **stable** if we have $\boxed{f_y(t, y) < 0}$ for all $t \in [t_0, T]$, $y \in \mathbb{R}$.

2 System of ODEs, higher order ODEs

We want to find n functions $y_1(t), \ldots, y_n(t)$ for $t \in [t_0, T]$ satisfying the differential equations

$$\begin{aligned} y_1'(t) &= f_1\left(t, y_1(t), \ldots, y_n(t)\right) \\ &\vdots \\ y_n'(t) &= f_n\left(t, y_1(t), \ldots, y_n(t)\right) \end{aligned}$$

and the initial conditions $y_1(t_0) = y_1^{(0)}, \ldots, y_n(t_0) = y_n^{(0)}$.

We use vector notation: E.g., for $n = 2$ we want to find $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ such that

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} f_1\left(t, y_1(t), y_2(t)\right) \\ f_2\left(t, y_1(t), y_2(t)\right) \end{bmatrix}, \qquad \begin{bmatrix} y_1(t_0) \\ y_2(t_0) \end{bmatrix} = \begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \end{bmatrix}$$

$$\vec{y}'(t) = \vec{f}\left(t, \vec{y}(t)\right), \qquad \vec{y}(t_0) = \vec{y}^{(0)}$$

We will omit the vector arrows from now on.

We denote by $D_y f(t, y)$ the Jacobian of $f(t, y)$ with respect to y:

$$D_y f(t,y) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix}$$

It is important for the behavior of the differential equation.

Theorem 2.1. *Assume that $f(t, y)$ and $D_y f(t, y)$ are continuous for $t \in [t_0, T]$, $y \in \mathbb{R}^n$. For a given initial value $y^{(0)}$ there is a unique solution $y(t)$ of the initial value problem. Either the solution exists for all $t \in [0, T]$, or it only exists on a smaller interval $[t_0, t_*)$ with $t_0 < t_* < T$.*

We can solve the IVP in Matlab with **`ode45`**: For $n = 2$ we use

```
f = @(t,y) [ ... ; ... ]       % define function f(t,y) using t, y(1), y(2)
[ts,ys] = ode45(f,[t0,T],y0);  % find column vector ts, array ys with values of solution
ys(end,:)                      % values y1, y2 at final time T
plot(ts,ys(:,1))               % plot solution y1(t)
```

2nd order ODE

So far the differential equations only contained the first derivative $y'(t)$. But in many applications (e.g. Newton's law) we have differential equations containing $y''(t)$. We then need initial conditions for $y(t_0)$ and $y'(t_0)$.

We can **rewrite this as a first order system:** Let $y_1(t) := y(t)$ and $y_2(t) := y'(t)$. Then we have $y_1' = y_2$ and $y_2' = \cdots$ where we solve the 2nd order ODE for y''.

Example: Find a function $y(t)$ for $t \in [0, 4]$ such that

$$y''(t) - y'(t) + 3y(t) = t \tag{1}$$

$$y(0) = 1, \quad y'(0) = -2 \tag{2}$$

This gives the first order system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ t + y_2 - 3y_1 \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \tag{3}$$

Numerical solution in Matlab: Print out $y(T)$ and plot the function $y(t)$

```
f = @(t,y) [y(2); t+y(2)-3*y(1)];   % define function f(t,y)
[ts,ys] = ode45(f,[0,4],[-1;2]);    % solve IVP for t from 0 to 4, initial value [-1;2]
finalval = ys(end,1)                % value of y1 at final time T
plot(ts,ys(:,1));                   % plot solution y1(t)
```

3 Euler method

Consider a first order system of ODEs: We want to find $y(t)$ for $t \in [t_0, T]$ such that

$$y'(t) = f\left(t, y(t)\right), \qquad y(t_0) = y^{(0)}$$

For the **Euler method** we divide the interval $[t_0, T]$ into N subintervals of equal length $h = (T - t_0)/N$ (we can also use subintervals of different length). Let $t_j = t_0 + jh$. We then want to find approximations $y^{(1)}, \ldots, y^{(N)}$ for $y(t_j)$.

- start at the initial value $t_0, y^{(0)}$
- for $k = 0, \ldots, N-1$ do
  $s := f(t_k, y^{(k)})$
  $y^{(k+1)} := y^{(k)} + hs$
  $t_{k+1} := t_k + h$

Example: Consider the Initial Value Problem (1), (2). Use 2 steps of the Euler method with $h = \frac{1}{2}$ to find approximations for $y(1)$ and $y'(1)$.
We use $y_1(t) := y(t)$ and $y_2(t) = y'(t)$ and obtain the first order ODE (3).

Step 1: $s = f(t_0, y^{(0)}) = f\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $y^{(1)} = y^{(0)} + h \cdot s = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4.5 \end{bmatrix}$

Step 2: $s = f(t_1, y^{(1)}) = f\left(\frac{1}{2}, \begin{bmatrix} 0 \\ 4.5 \end{bmatrix}\right) = \begin{bmatrix} 4.5 \\ 5 \end{bmatrix}$, $y^{(2)} = y^{(1)} + h \cdot s = \begin{bmatrix} 0 \\ 4.5 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 4.5 \\ 5 \end{bmatrix} = \begin{bmatrix} 2.25 \\ 7 \end{bmatrix}$

This gives $y(1) \approx 2.25$ and $y'(1) \approx 7$.

Errors for Euler method

We consider the case $n = 1$. At time t_k the Euler method gives an approximation y_k for the exact value $y(t_k)$. We denote the **error** by

$$e_k := y_k - y(t_k)$$

By Taylor's theorem we have with a remainder term $r_k = \frac{1}{2} y''(\tau_k) h^2$

$$\begin{aligned} y(t_{k+1}) &= y(t_k) + h \cdot \overbrace{f\,(t_k, y(t_k))}^{y'(t_k)} + r_k \\ y_{k+1} &= y_k \quad\; + h \cdot f(t_k, y_k) \end{aligned}$$

The second equation is just the definition of the Euler approximation y_{k+1}. Subtracting the first from the second equation gives

$$e_{k+1} = e_k + h \cdot [f(t_k, y_k) - f(t_k, y(t_k))] - r_k$$

Using the mean value theorem for $g(y) := f(t_k, y)$

$$f(t_k, y_k) - f(t_k, y(t_k)) = f_y(t_k, \eta_k) \cdot [y_k - y(t_k)],$$

hence the new error is

$$\boxed{e_{k+1} = \underbrace{[1 + h f_y(t_k, \eta_k)]}_{a_k} e_k - r_k}$$

with the **amplification factor** $a_k = 1 + h f_y(t_k, \eta_k)$ and the local **truncation error** $r_k = \frac{1}{2} y''(\tau_k) h^2$**.**

For an unstable ODE we have $f_y(t, y) > 0$ and hence $a_k > 1$.

For a stable ODE we have $f_y(t, y) < 0$ and hence $a_k < 1$. However, in this case we want $|a_k| < 1$, i.e.,

$$-1 < 1 + h f_y(t, y) < 1$$

The right inequality is true for any $h > 0$. The left inequality is true if the following **stability condition** holds:

$$\boxed{h < \frac{2}{-f_y}}$$

(1) General case:

We assume

$$\begin{aligned} |f_y(t,y)| &\le C_1 \quad \text{for } t \in [t_0,T],\ y \in \mathbb{R} \\ \left|y''(t)\right| &\le C_2 \quad \text{for } t \in [t_0,T] \end{aligned}$$

Then we get bounds $|a_k| \le A$ for the amplification factor and $|r_k| \le R$ for the local truncation error:

$$\begin{aligned} |a_k| &= |1 + h f_y(t_k,\eta_k)| \le 1 + hC_1 =: A \\ |r_k| &= \left|\frac{1}{2} y''(\tau_k) h^2\right| \le \frac{C_2}{2} h^2 =: R \end{aligned}$$

yielding

$$|e_{k+1}| \le A\,|e_k| + R$$

Since $|e_0| = 0$ we obtain

$$\begin{aligned} |e_1| &\le R \\ |e_2| &\le AR + R = (1+A)\,R \\ |e_3| &\le A(1+A)R + R = \left(1 + A + A^2\right) R \\ &\ \vdots \\ |e_k| &\le \left(1 + A + \cdots + A^{k-1}\right) R \end{aligned} \tag{4}$$

We have for the geometric series

$$1 + A + \cdots + A^{k-1} = \frac{A^k - 1}{A - 1} \le \frac{A^k}{A-1} = \frac{(1+hC_1)^k}{hC_1}$$

The function e^x satisfies $1 + x \le e^x$, hence with $x = hC_1$ we get $1 + hC_1 \le e^{hC_1}$. Using this and $R = \frac{C_2}{2}h^2$ in (4) gives the **error bound**

$$\boxed{|y_k - y(t_k)| \le \frac{C_2}{2C_1} e^{C_1(t_k - t_0)} h}$$

This shows:

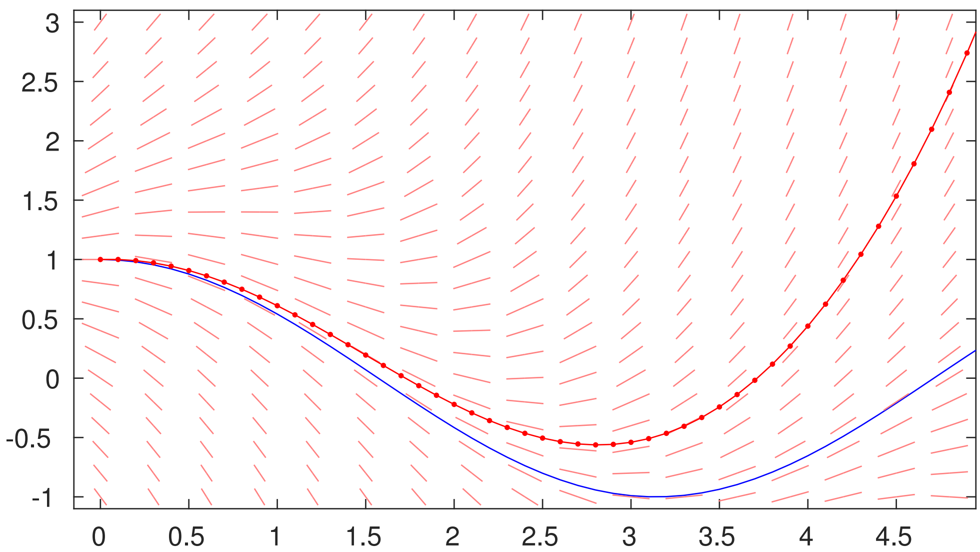
- if we keep taking Euler steps with a fixed value h for $t \to \infty$ the error can increase exponentially. This is not surprising: For an unstable ODE any tiny initial error can cause an exponentially increasing error for $t \to \infty$.
- if we only want to find the solution for $t \in [t_0,T]$: We use $h = \frac{T-t_0}{N}$ and obtain errors bounded by $ch = c'N^{-1}$: **The Euler method is a method of order 1**.

Example: The initial value problem

$$y' = y - \sin t - \cos t, \qquad y(0) = 1$$

has the solution $y(t) = \cos t$. Here $f_y(t,y) = 1 > 0$. We use the Euler method with $h = 0.1$:

```
f = @(t,y) y-sin(t)-cos(t)
tv = 0:.1:5; plot(tv,cos(tv),'b'); hold on     % plot exact solution
dirfield(f,0:.3:5,-1:.2:3);
[ts,ys] = Euler(f,[0,5],1,50);                 % use 50 steps of size 5/50
plot(ts,ys,'r.-'); hold off
```



We see that the Euler values go exponentially to $+\infty$ as t gets larger than 4, whereas the exact solution $y(t) = \cos t$ stays bounded.

(2) Special case: Stable ODE where h satisfies stability condition:

We assume $f_y(t, y) < 0$: We have $C_1 \geq C_0 > 0$ such that

$$-C_1 \leq f_y(t, y) \leq -C_0 \quad \text{for } t \in [t_0, T],\ y \in \mathbb{R}$$
$$\left|y''(t)\right| \leq C_2 \quad \text{for } t \in [t_0, T]$$

We now want to have an amplification factor with $|a_k| \leq 1 - C_0 h$, i.e.,

$$-(1 - C_0 h) \leq 1 + h f_y(t_k, \eta_k) \leq 1 - C_0 h$$

The right inequality holds for any $h > 0$. We have $1 - hC_1 \leq 1 + h f_y(t_k, \eta_k)$, therefore the left inequality holds if $-(1 - C_0 h) \leq 1 - hC_1$ or

$$\boxed{h \leq \frac{2}{C_0 + C_1}} \tag{5}$$

If h satisfies this stability condition we have $|a_k| \leq 1 - C_0 h =: A < 1$, hence

$$1 + A + \cdots + A^{k-1} = \frac{1 - A^k}{1 - A} \leq \frac{1}{1 - A}$$

and (4) now gives

$$\boxed{|y_k - y(t_k)| \leq \frac{C_2}{2C_0} h}$$

This shows:

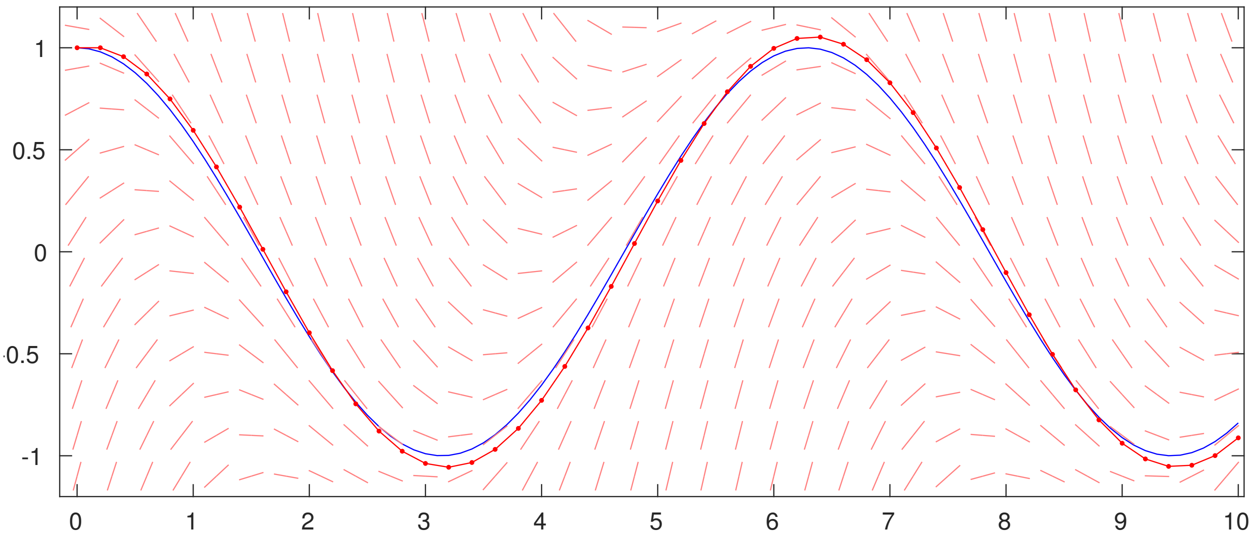
- if we keep taking Euler steps with a fixed value h for $t \to \infty$ the error is bounded by Ch with a fixed constant C.

Example: The initial value problem

$$y' = -y - \sin t + \cos t, \qquad y(0) = 1$$

has the solution $y(t) = \cos t$. Here $f_y(t, y) = -1 < 0$. We use the Euler method with $h = 0.2$:

```
f = @(t,y) -y-sin(t)+cos(t)
tv = 0:.1:10; plot(tv,cos(tv),'b'); hold on   % plot exact solution
dirfield(f,0:.3:10,-1.1:.2:1.1);
[ts,ys] = Euler(f,[0,10],1,50);               % use 50 steps of size 10/50
plot(ts,ys,'r.-'); hold off
```



Here we have an error of size Ch, but the error stays bounded as $t \to \infty$.

4 Improved Euler method (aka RK2 method)

For the Euler method we obtained a **local truncation error** r_k satisfying $|r_k| \le ch^2$. Since we use $N = (T - t_0)/h$ steps to get from t_0 to the final time T, we obtained **the error at time** T was bounded by Ch^1: The Euler method is a **method of order 1**. This means that in order to improve the error by a factor of 10, we need to use 10 times as many steps. If we want to achieve an error of size 10^{-8} we may need of the order of 10^8 steps (assuming e.g. a stable problem with C_2 and C_0 of about 1).

We would like to have a **method of order 2**, i.e., the error at time T is bounded by Ch^2. This means that the local truncation error should satisfy $|r_k| \le ch^3$. How can we do this? We need more than one evaluation of the direction field per step.

We start at the initial point (t_0, y_0). We take a step of size h to $t_1 = h$ and want to find an approximation y_1 for $y(t_1)$. We know that $y'(t) = f(t, y(t))$, so by the fundamental theorem of calculus we have

$$y(t_1) = y_0 + \int_{t_0}^{t_1} f(t, y(t))\, dt$$

Let $g(t) := f(t, y(t))$. Then we have to approximate the integral $I = \displaystyle\int_{t_0}^{t_1} g(t)dt$ with an error$\le ch^3$. One way to do this is to use the trapezoid rule: Recall that on an interval of size h we have $\left|Q^{\text{Trap}} - I\right| \le \frac{h^3}{12} \max |g''|$. Therefore we want to use

$$I \approx Q^{\text{Trap}} = \frac{h}{2}\left[g(t_0) + g(t_1)\right] = \frac{h}{2}\Big[f(t_0, y_0) + f(t_1, \underbrace{y(t_1)}_{\textbf{?}})\Big]$$

But we cannot evaluate the second term $f(t_1, y(t_1))$ since we don't know $y(t_1)$. So we use the best approximation we have: the Euler approximation $y^{\text{Euler}} = y_0 + h \cdot f(t_0, y_0)$. Since $\left|y^{\text{Euler}} - y(t_1)\right| \le Ch^2$ we get from the mean value theorem

$$\left|f(t_1, y^{\text{Euler}}) - f(t_1, y(t_1))\right| = \left|\frac{\partial f}{\partial y}(t_1, \eta) \cdot \left(y^{\text{Euler}} - y(t_1)\right)\right| \le C_1 \cdot Ch^2$$

We now approximate $y(t_1)$ by

$$\boxed{y_1 := y_0 + \frac{h}{2}\Big[f(t_0, y_0) + f(t_1, y^{\text{Euler}})\Big]} \tag{6}$$

and obtain for the local truncation error

$$|y_1 - y(t_1)| \le Ch^3$$

since the error of the trapezoid rule is bounded by ch^3, and replacing $y(t_1)$ by y^{Euler} causes an additional error $\frac{h}{2}C_1Ch^2$.

The iteration (6) gives the **Improved Euler method**: we divide the interval $[t_0, T]$ into N subintervals of equal length $h = (T - t_0)/N$ (we can also use subintervals of different length). Let $t_j = t_0 + jh$. We then want to find approximations $y^{(1)}, \ldots, y^{(N)}$ for $y(t_j)$.

- start at the initial value $t_0, y^{(0)}$
- for $k = 0, \ldots, N-1$ do
  $s^{(1)} := f(t_k, y^{(k)})$
  $y^E := y^{(k)} + hs^{(1)}$
  $s^{(2)} := f(t_k + h, y^E)$
  $y^{(k+1)} := y^{(k)} + \frac{1}{2}\left[s^{(1)} + s^{(2)}\right]$
  $t_{k+1} := t_k + h$

The local truncation error of the improved Euler method is of order $O(h^3)$. Hence the error at a time $t = T$ is of order $O(h^2) = O(N^{-2})$: **The improved Euler method is a method of order 2**. Note that we use two evaluations of the function f per step: $s^{(1)} = f(t_k, y^{(k)})$ and $s^{(2)} = f(t_k + h, y^E)$.

Example: Consider the Initial Value Problem (1), (2). Use 1 step of the Improved Euler method with $h = \frac{1}{2}$ to find approximations for $y(\frac{1}{2})$ and $y'(\frac{1}{2})$.
We use $y_1(t) := y(t)$ and $y_2(t) = y'(t)$ and obtain the first order ODE (3).
$s^{(1)} = f(t_0, y^{(0)}) = f\left(0, \left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}2\\5\end{smallmatrix}\right]$, $y^E = y^{(0)} + h \cdot s = \left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right] + \frac{1}{2}\cdot\left[\begin{smallmatrix}2\\5\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\4.5\end{smallmatrix}\right]$
$s^{(2)} = f(t_1, y^E) = f\left(\frac{1}{2}, \left[\begin{smallmatrix}0\\4.5\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}4.5\\5\end{smallmatrix}\right]$, $y^{(1)} = y^{(0)} + \dfrac{h}{2}\left(s^{(1)} + s^{(2)}\right) = \left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right] + \frac{1}{4}\left(\left[\begin{smallmatrix}2\\5\end{smallmatrix}\right] + \left[\begin{smallmatrix}4.5\\5\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}0.625\\4.5\end{smallmatrix}\right]$
This gives $y(\frac{1}{2}) \approx 0.625$ and $y'(\frac{1}{2}) \approx 4.5$.

5 Stiff ODE and ode15s

Consider a "very stable" ODE where $f_y(t, y)$ is very negative. Then the Euler method only works if the step size h satisfies the stability condition $h < \dfrac{2}{-f_y}$. This can force use to use very tiny steps even if the solution $y(t)$ is almost constant. This is called a **stiff ODE**. In this case `ode45` uses many tiny steps and takes a long time.

Example: A flame propagation model gives the following IVP:

$$y' = y^2 - y^3 \qquad \text{for}\, t \in [0, \frac{2}{\delta}]$$
$$y(0) = \delta$$

Here δ is very small, e.g., $\delta = 10^{-4}$. In this case we want to solve the problem for $t \in [0, \frac{2}{\delta}] = [0, 2 \cdot 10^4]$. The solution approaches $y = 1$. But there the problem becomes stiff: We have near $y = 1$ that $f_y(t, y) = 2y - 3y^2 \approx -1$, so the stability condition for the Euler method requires $h < \dfrac{2}{-f_y} = 2$. This means that we need $N = 10^4$ steps of size $h = 2$ to get from $t_0 = 0$ to $T = 2/\delta$, despite the fact that the solution is almost constant for most of $[0, T]$.

The adaptive method **`ode45`** (with default settings) also requires about 10^4 steps:

```
delta = 1e-4;
f = @(t,y) y^2-y^3;
y0 = delta;
t0 = 0;  T = 2/delta;
[ts,ys] = ode45(f,[0,T],y0);
length(ts)                         % print number of steps
```

This prints out 12113 for the number of steps.

Matlab has a special ode solver **ode15s** for stiff ODEs: We try this for our problem

```
[ts,ys] = ode15s(f,[0,T],y0);
length(ts)                         % print number of steps
```

and get 108 for the number of steps.

6 Backward Euler method (aka implicit Euler method)

For stable problems the Euler method gives magnification factors $|a_k| = |1 + hf_y| < 1$ for small h, but $|a_k| = |1 + hf_y| > 1$ for large h.

If I look at a problem with $f_y < 0$ from right to left with decreasing t, then an Euler method in decreasing t direction always has a magnification factor $|a| > 1$, for any step size $h > 0$.

This suggests to use a "**backward Euler step**":

At time t_k we have the value $y^{(k)}$.

For time $t_{k+1} = t_k + h$ we want to find a value $y^{(k+1)}$ such that an Euler step to the left takes us to t_k and $y^{(k)}$:

Find $y^{(k+1)}$ such that

$$\boxed{y^{(k+1)} - h \cdot f\left(t_{k+1}, y^{(k+1)}\right) = y^{(k)}} \tag{7}$$

Note that the unknown vector $y^{(k+1)}$ occurs also inside the function f. If the function $f(t, y)$ is linear in y this gives linear equations for y. If the function $f(t, y)$ is nonlinear in y this gives nonlinear equations for y. We can e.g. use 1 or 2 steps of the Newton method (note that we have a local truncation error of size $O(h^2)$).

Example: Consider the Initial Value Problem (1), (2). Use 1 step of the backward Euler method with $h = \frac{1}{2}$ to find approximations for $y(\frac{1}{2})$ and $y'(\frac{1}{2})$.
We use $y_1(t) := y(t)$ and $y_2(t) = y'(t)$ and obtain the first order ODE (3). Note that the function $f(t, y) = \begin{bmatrix} y_2 \\ t + y_2 - 3y_1 \end{bmatrix}$ depends linearly on y_1, y_2.

We want to find a vector $y^{(1)} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ such that

$$\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] - h \cdot f(t_1, \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]) = y^{(0)}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} y_2 \\ \frac{1}{2} + y_2 - 3y_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

This gives the linear system $\begin{bmatrix} y_1 - \frac{1}{2}y_2 \\ \frac{3}{2}y_1 + \frac{1}{2}y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2.25 \end{bmatrix}$ which has the solution $y^{(1)} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 3 \end{bmatrix}$. This gives $y(\frac{1}{2}) \approx 0.5$ and $y'(\frac{1}{2}) \approx 3$.

Claim: Let $n = 1$. For a stable problem with $f_y(t, y) < 0$ the nonlinear equation has a unique solution.
Proof: The left hand side $F(y_{k+1}) := y_{k+1} - h \cdot f(t_{k+1}, y_{k+1})$ is strictly increasing for increasing y_{k+1}, with $F(y) \to -\infty$ for $y \to -\infty$ and $F(y) \to \infty$ for $y \to \infty$.