Example for Gaussian Elimination with Pivoting

Solve the linear system $\begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$. Use the **pivot candidate with the largest absolute value.**

Gaussian Elimination for matrix A:

Initialize L, U, p. Then select pivot for column 1:

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Move pivot for column 1 in position: interchange rows 1 and 4

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Elimination in column 1. Then select pivot for column 2

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Move pivot for column 2 in position: interchange rows 2 and 3

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Elimination in column 2. Then select pivot for column 3

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Move pivot for column 3 in position: interchange rows 3 and 4

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{5} & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \end{bmatrix}, \quad p = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

Elimination in column 3

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{5} & \frac{1}{5} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{3}{5} \end{bmatrix}, \quad p = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

The last pivot is $\frac{3}{5}$. This is nonzero, so the algorithm succeeded. Therefore A is **nonsingular**.

Finally put 1's on the diagonal of L, yielding $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & \frac{3}{5} & \frac{1}{5} & 1 \end{bmatrix}$.

Given b, use L, U, p to solve linear system:

Solve $Ly = \begin{bmatrix} b_{p_1} \\ \vdots \\ b_{p_n} \end{bmatrix}$ by forward substitution:

$$\text{Solving} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & \frac{3}{5} & \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{gives} \quad y = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

Solve $Ux = y$ by back substitution:

$$\text{Solving} \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 3 \end{bmatrix} \quad \text{gives} \quad x = \begin{bmatrix} 1 \\ 2 \\ -5 \\ 5 \end{bmatrix}$$

How to do this in Matlab:

```
>> A = [0 0 1 1;-1 1 0 0; 1 3 1 0; 2 1 1 1]
A =
     0     0     1     1
    -1     1     0     0
     1     3     1     0
     2     1     1     1
>> [L,U,p] = lu(A,'vector')
L =
            1             0             0             0
          0.5             1             0             0
            0             0             1             0
         -0.5           0.6           0.2             1
U =
            2             1             1             1
            0           2.5           0.5          -0.5
            0             0             1             1
            0             0             0           0.6
p =
     4     3     1     2
>> b = [0;1;2;4];
>> y = L\b(p)
y =
     4
     0
     0
     3
>> x = U\y
x =
            1
            2
           -5
            5
```