AMSC/CMSC 460 Midterm Exam 2 - Solutions

Tuesday, April 10th, 2018

You have 75 minutes to complete this exam. **No** Calculators or cheat sheets are allowed. Submit each problem on a separate sheet. Show all work and explain your answers.

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1. Consider the data points

x	-1	0	1	2
$f(x)$	.5	1	2	4

(a) **[15pts]** Write the Lagrange form of the cubic interpolating polynomial $Q_3(x)$ for the above data.

Solution: The Lagrange form of $Q_3(x)$ is given by

$$Q_3(x) = \sum_{i=0}^{3} f(x_i)\ell_i(x), \quad \ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}} \frac{x - x_j}{x_i - x_j}$$

writing this out explicitly interms of the above data gives

$$Q_3(x) = -\frac{x(x-1)(x-2)}{12} + \frac{(x+1)(x-1)(x-2)}{2} - (x+1)x(x-2) + 2\frac{(x+1)x(x-1)}{3}$$

(b) **[20pts]** Write the Newton form of that same interpolating polynomial. Be sure to compute all divided differences.

Solution: We begin by writing out the divided difference table

x	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_{i-1}, x_i, x_{i+1}]$	$f[x_0, x_1, x_2, x_3]$
-1	$1/2$	$1/2$	$1/4$	$1/12$
0	1	1	$1/2$	
1	2	2		
2	4			

The Newton form of $Q_3(x)$ is given by

$$Q_3(x) = \sum_{i=0}^{3} f[x_0, \ldots, x_i] \prod_{j=0}^{i-1}(x - x_j),$$

which, using the values computed in the divided difference table, gives

$$Q_3(x) = \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{4}x(x+1) + \frac{1}{12}x(x+1)(x-1).$$

(c) **[15pts]** Using the fact that this sample was taken from the function $f(x) = 2^x$, write an expression for the interpolation error $e(x) = f(x) - Q_3(x)$.

Solution: We know from class that for each $x \in [-1, 2]$, there exists ξ_x belonging to $[-1, 2]$ such that the error between $f(x) = 2^x$ and $Q_3(x)$ is given by

$$e(x) = f(x) - Q_3(x) = \frac{1}{4!}f^{(4)}(\xi_x)(x+1)x(x-1)(x-2) = \frac{(\ln(2))^4}{24}2^{\xi_x}(x+1)x(x-1)(x-2)$$

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2. (a) **[10pts]** Write down a linear system for the coefficients a_0, a_1, a_2 which define the quadratic polynomial $Q(x) = a_0 + a_1 x + a_2 x^2$ that satisfies the conditions $Q(x_0) = y_0$, $Q'(x_1) = y_1$, $Q'(x_2) = y_2$.

Solution: The system of equations that satisfy these constraints is given by

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\ a_1 + 2a_2 x_1 &= y_1 \\ a_1 + 2a_2 x_2 &= y_2. \end{aligned}$$

(b) **[20pts]** Assuming $x_1 \neq x_2$ solve this system for a_0, a_1, a_2. (Hint: Solve for a_1 and a_2 first).

Solution: The coefficients a_1 and a_2 satisfy a 2×2 linear sytem and can be solved first to obtain

$$a_1 = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}, \qquad a_2 = \frac{1}{2} \frac{y_2 - y_1}{x_2 - x_1}$$

we can then substitute the values for a_1 and a_2 to obtain a_0

$$a_0 = y_0 - a_1 x_0 - a_2 x_0^2 = y_0 - \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1} x_0 - \frac{1}{2} \frac{y_2 - y_1}{x_2 - x_1} x_0^2.$$

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3. Consider the weighted inner product

$$\langle f, g \rangle_{e^{-x^2}} = \int_0^\infty f(x) g(x) e^{-x^2} \mathrm{d}x.$$

(a) **[15pts]** Find the first two polynomials $P_0(x)$ and $P_1(x)$ which are orthogonal with respect to the above inner product.

Solution: We carry out the Gram-Schmidt orthogonalization procedure for this inner product. This gives

$$P_0(x) = 1, \quad P_1(x) = x - \frac{\int_0^\infty x e^{-x^2} \mathrm{d}x}{\int_0^\infty e^{-x^2} \mathrm{d}x} P_0(x) = x - \frac{1}{\sqrt{\pi}}.$$

(b) **[15pts]** Find the corresponding weighted norm of each polynomial.

Solution: The norms are

$$\|P_0\|_{e^{-x^2}}^2 = \int_0^\infty e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

and

$$\begin{aligned} \|P_1\|_{e^{-x^2}}^2 &= \int_0^\infty \left(x - \frac{1}{\sqrt{\pi}} \right)^2 e^{-x^2} \mathrm{d}x \\ &= \int_0^\infty \left(x^2 - \frac{2}{\sqrt{\pi}} x + \frac{1}{\pi} \right) e^{-x^2} \mathrm{d}x \\ &= \frac{\sqrt{\pi}}{4} - \frac{1}{\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}} = \frac{\pi - 2}{4\sqrt{\pi}} \end{aligned}$$

(c) **[20pts]** Use these orthogonal polynomials to find the polynomial $Q_1(x)$ of degree ≤ 1 that minimizes

$$\int_0^\infty (x^2 - Q_1(x))^2 e^{-x^2} \mathrm{d}x.$$

Solution: We know that the optimal polynomial is given by

$$Q_1(x) = c_0 P_0(x) + c_1 P_1(x),$$

where c_0 and c_1 satisfy

$$c_0 = \frac{\int_0^\infty x^2 P_0(x) e^{-x^2} \mathrm{d}x}{\|P_0\|^2_{e^{-x^2}}} = \frac{\sqrt{\pi}/4}{\sqrt{\pi}/2} = \frac{1}{2}$$

and

$$c_1 = \frac{\int_0^\infty x^2 P_1(x) e^{-x^2} \mathrm{d}x}{\|P_1\|^2_{e^{-x^2}}} = \frac{4\sqrt{\pi}\int_0^\infty \left(x^3 - \frac{x^2}{\sqrt{\pi}}\right) e^{-x^2} \mathrm{d}x}{\pi - 2} = \frac{\sqrt{\pi}}{\pi - 2}.$$

Therefore $Q_1(x)$ is

$$Q_1(x) = \frac{1}{2} - \frac{1}{\pi - 2} + \frac{\sqrt{\pi}}{\pi - 2} x.$$

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4. **[20pts]** Suppose you want to fit the following trigonometric polynomial

$$h(x) = a\cos(x) + b\sin(x) + c\cos(2x),$$

to the data (x_i, y_i), $i = 0, 4$, given by $(0,1), (0,0), (\pi/2, 1), (\pi/2, 2), (\pi, 1)$ by finding a, b, c such that

$$\sum_{i=0}^{4} (h(x_i) - y_i)^2,$$

is minimized. Write down, *but do not solve*, the normal equations associated to this least-squares problem. (Simplify all matrix products involved).

Solution: We can start by writing least squares problem in terms of a matrix $\mathbf{A}$, and vectors $\mathbf{x} = (a, b, c)^\top$ and $\mathbf{y} = (y_0, y_1, y_2, y_3, y_4)^\top$. We have

$$\sum_{i=0}^{4} (h(x_i) - y_i)^2 = \|\mathbf{A}\mathbf{x} - \mathbf{y}\|^2$$

where

$$\mathbf{A} = \begin{bmatrix} \cos x_0 & \sin x_0 & \cos 2x_0 \\ \cos x_1 & \sin x_1 & \cos 2x_1 \\ \cos x_2 & \sin x_2 & \cos 2x_2 \\ \cos x_3 & \sin x_3 & \cos 2x_3 \\ \cos x_4 & \sin x_4 & \cos 2x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

The normal equations associated with this least square system are

$$\mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{y}.$$

Since

$$\mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

and

$$\mathbf{A}^\top \mathbf{y} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix},$$

the normal equations are

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}.$$