Numerical Differential Equations
Lecture 1
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These notes will supplement the lectures on numerical differential equations. Roughly speaking, they are a distillation of Iserles' book on the topic.

1 Review of differential equations

Definition 1. An *initial value problem* (IVP) is a problem in which $f(t, y) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $y_0 \in \mathbb{R}^n$ are known and the goal is to find a differentiable function $y(t) : (0, T) \to \mathbb{R}^n$ such that

$$\frac{d}{dt}y(t) = f(t, y(t)) \quad t \in [0, T_+) \tag{1}$$

$$y(0) = y_0. \tag{2}$$

Remark 1. As mentioned briefly in class, we can also solve the equations backwards in time, but it isn't really any different or harder, so let me just ignore this right now.

The number of differential equations which can be solved by hand is extremely short, however, let's do a few examples to refresh (?) ourselves about what the behavior looks like.

Example 1. The simplest example is when f does not depend on y:

$$\frac{d}{dt}y(t) = f(t) \tag{3}$$

$$y(0) = y_0. \tag{4}$$

In this case, we just integrate (3) and use the fundamental theorem of calculus:

$$y(t) - y(0) = \int_0^t \frac{d}{ds}y(s)ds = \int_0^t f(s)ds, \tag{5}$$

and so the solution is

$$y(t) = y_0 + \int_0^t f(s)ds.$$

Example 2. The next simplest example is when $f(t, y) = \alpha y(t) + g(t)$ for $\alpha \in \mathbb{R}$ ($\alpha \in \mathbb{C}$ also works the same way and is also important) and g known. That is, the IVP

$$\frac{d}{dt}y(t) = \alpha y(t) + g(t). \tag{6}$$

In this case, we cannot just integrate the equation to get the answer. Indeed, doing this would give the

$$y(t) = y(0) + \alpha \int_0^t y(s)ds + \int_0^t g(s)ds. \tag{7}$$

and we haven't succeeded in solving for y. The trick for solving this is called *integrating factors.* We learned in intro calculus that

$$\frac{d}{dt}e^{\alpha t} = \alpha e^{\alpha t}, \tag{8}$$

and so multiplying (1) by $e^{-\alpha t}$ and using the (backwards) product rule:

$$e^{-\alpha t}\frac{d}{dt}y(t) - \alpha e^{-\alpha t}y(t) = e^{-\alpha t}g(t) \tag{9}$$

$$\frac{d}{dt}\left(e^{-\alpha t}y(t)\right) = e^{-\alpha t}g(t). \tag{10}$$

This we can now integrate using the fundamental theorem of calculus to get:

$$e^{-\alpha t}y(t) - e^{-\alpha 0}y(0) = \int_0^t e^{-\alpha s}g(s)ds, \tag{11}$$

which re-arranges to

$$y(t) = e^{\alpha t}y_0 + \int_0^t e^{\alpha(t-s)}g(s)ds. \tag{12}$$

Example 3. The next example is a little trickier. Consider the IVP:

$$\frac{d}{dt}y(t) = (y(t))^2 \tag{13}$$

$$y(0) = y_0 \in (0, \infty). \tag{14}$$

The way to solve this is a trick called *separation of variables.* We divide by $y(t)$,

$$\frac{1}{(y(t))^2}\frac{d}{dt}y(t) = 1. \tag{15}$$

Then, we observe from the chain rule that this implies

$$-\frac{d}{dt}\left(\frac{1}{(y(t))}\right) = 1. \tag{16}$$

Then, we integrate using the fundamental theorem of calculus:

$$\frac{1}{y(0)} - \frac{1}{y(t)} = t. \tag{17}$$

Re-arranging gives

$$y(t) = \frac{y_0}{1 - y_0 t}. \tag{18}$$

This is very interesting because this solution has a *vertical asymptote* at $t = y_0^{-1}$. This explains why I insisted on specifying a T_+ in definition 1: even if f is a smooth function, the solution to the IVP may not last forever.

This last main theorem you only have to be vaguely familiar with. It is simply an assertion that the IVPs we are discussing are solvable. See an upper division differential equations class or perhaps Math 411 for a proof and a more in-depth discussion.

Theorem 1 (Cauchy-Lipschitz theorem)**.** *Let f be a smooth function. Then, for all $y_0 \in \mathbb{R}^n$, $\exists!$ $0 < T_+ \leq \infty$ and smooth function $y : [0, T_+) \to \mathbb{R}^n$ which solves*

$$\frac{d}{dt} y(t) = f(t, y(t)) \quad t \in [0, T_+) \tag{19}$$

$$y(0) = y_0. \tag{20}$$

Moreover, either $T_+ = \infty$ or $\lim_{t \nearrow T_+} \|y(t)\| = \infty$.

Remark 2. That latter statement asserts that nothing can go wrong *other* than the solution escaping to infinity in finite time as in Example 3.

2 Intro to the numerical solution of differential equations

We are concerned now with using a computer[1] to approximate the solution given by Theorem 1 for $t \in [0, T]$ (where $T < T_+$). Set a step-size $h > 0$ and define $t_n = hn$. The goal is approximate the solution at these discrete times. That is, to find an approximation $y_{n,h}$ such that

$$y_{n,h} \approx y(t_n). \tag{21}$$

We often omit the h and simply write y_n (but its important to remember its there!). The key observation is that from the fundamental theorem of calculus, the solution from Theorem 1 satisfies the following for all $t_{n+1} \leq T$:

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s)) ds. \tag{22}$$

Hence, our goal is to numerically approximate the integral on the right-hand side. The simplest idea would be to use a left-Riemann sum and make the approximation:

$$\int_{t_n}^{t_{n+1}} f(s, y(s)) ds = h f(t_n, y(t_n)) + O(h^2). \tag{23}$$

Recall that this is proved via Taylor expansion of $g(s) = f(s, y(s))$ around the point $s = t_n$. With that observation in hand, this leads us to the idea that we can choose our approximation via the recursion scheme:

$$y_{n+1,h} = y_{n,h} + h f(t_n, y_{n,h}) \tag{24}$$

$$y_{0,h} = y_0. \tag{25}$$

This recursion scheme is called *(forward) Euler's method* (pronounced 'oiler'). The next question we want to ask is: does this actually work?

Definition 2. A numerical differential equation method is *convergent of order p* if

$$\max_{0 \leq n \leq T/h} \|y_{n,h} - y(t_n)\| = O(h^p). \tag{26}$$

Rigorous analysis of ODE solvers is tricky business, and mostly beyond the scope of the course. However, let's see one proof of this type. I want you to be able to do the first part of the proof, but the second part might be too hard for people who have not taken Math 410.

[1] Though solving differential equations is so important, many of these techniques were used first to compute by hand!

Theorem 2. *Suppose f satisfies*

$$\|f(t,y_1) - f(t,y_2)\| \le \lambda \|y_1 - y_2\| \quad \text{for some} \quad \lambda > 0. \tag{27}$$

Then Euler's method is convergent of order one.

Remark 3. Theorem 2 applies to all smooth f, not just those satisfying (27), but the proof is slightly more annoying to write down (especially if you haven't taken Math 410/411) in the more general case and it doesn't really make a difference for understanding numerical differential equations.

Proof. As noted above in (22) and (23), we have

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + O(h^2). \tag{28}$$

Hence, using also the definition of the scheme (24),

$$y(t_{n+1}) - y_{n+1,h} = y(t_n) - y_{n,h} + hf(t_n, y(t_n)) - hf(t_n, y_{n,h}) + O(h^2). \tag{29}$$

Define the error

$$e_{n,h} = y(t_n) - y_{n,h}. \tag{30}$$

Then, the above is equivalent to

$$e_{n+1,h} = e_{n,h} + hf(t_n, y(t_n)) - hf(t_n, y_{n,h}) + O(h^2).$$

Taking the norm of both sides and using the triangle inequality, we get the following for some $C > 0$ (by the definition of $O(h^2)$),

$$\begin{aligned} \|e_{n+1,h}\| &= \|e_{n,h} + hf(t_n, y(t_n)) - hf(t_n, y_{n,h}) + O(h^2)\| && (31) \\ &\le \|e_{n,h}\| + h\|f(t_n, y(t_n)) - hf(t_n, y_{n,h})\| + Ch^2. && (32) \end{aligned}$$

Then, (27) implies

$$\|e_{n+1,h}\| \le (1 + h\lambda)\|e_{n,h}\| + Ch^2. \tag{33}$$

I would like you to be able to derive (33) for similar numerical schemes. This is is an estimate of how quickly the error grows. Each time-step you add a new $O(h^2)$ error but you also multiplicatively increase the size of the error from the previous time-step (note $1 + h\lambda > 1$). I claim by induction that

$$\|e_{n,h}\| \le Ch^2 \left(\frac{(1 + h\lambda)^n - 1}{h\lambda} \right). \tag{34}$$

This is true for $n = 0$ because we get the initial condition exactly:

$$\|e_{0,h}\| = Ch^2 \left(\frac{(1 + h\lambda)^0 - 1}{h\lambda} \right) = 0. \tag{35}$$

If we assume that (34) holds for n, then we have from (33) that

$$\|e_{n+1,h}\| \le (1 + h\lambda)Ch^2 \left(\frac{(1 + h\lambda)^n - 1}{h\lambda} \right) + Ch^2 = Ch^2 \left(\frac{(1 + h\lambda)^{n+1} - 1}{h\lambda} \right), \tag{36}$$

and hence the (34) holds for all n by induction. Finally, since $e^x = \sum_{k=0}^{\infty} x^k/k!$, we have

$$1 + h\lambda \leq e^{h\lambda}, \tag{37}$$

and hence, (34) gives

$$\|e_{n,h}\| \leq \frac{Ch}{\lambda}\left(e^{hn\lambda} - 1\right).$$

Finally, $nh \leq T$ and hence

$$\|e_{n,h}\| \leq \frac{Ch}{\lambda}\left(e^{\lambda T} - 1\right). \tag{38}$$

The point here is that this latter inequality is of the form $\|e_{n,h}\| \leq C'h$, where I am denoting

$$C' = \frac{C}{\lambda}\left(e^{\lambda T} - 1\right), \tag{39}$$

where C' does not depend on h or n, only on T and λ quantities which are independent of the step-size. Hence, we have that

$$\|e_{n,h}\| = O(h), \tag{40}$$

for all n, so that

$$\max_{0 \leq n \leq T/h} \|e_{n,h}\| = O(h). \tag{41}$$

Hence, Definition 2 holds. □