Math 241 - Exam 1

Tuesday, Sept 26th, 2017

You have 50 minutes to complete this exam. Do not simplify unless indicated. Calculators are **not** allowed. Submit each of the five problems on a separate sheet. Partial credit will be given.

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(1) (a) [**8 pts**] Let $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ be two vectors. Find $Pr_{\mathbf{b}}\mathbf{a}$ and $Pr_{\mathbf{a}}\mathbf{b}$.

Partial Solution: Use the formula

$$Pr_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

You should obtain

$$Pr_{\mathbf{b}}\mathbf{a} = -\frac{10}{29}\mathbf{i} + \frac{15}{29}\mathbf{j} - \frac{20}{29}\mathbf{k}$$

and

$$Pr_{\mathbf{a}}\mathbf{b} = -\frac{5}{3}\mathbf{i} - \frac{5}{3}\mathbf{j} + \frac{5}{3}\mathbf{k}.$$

(b) [**12 pts**] Find the area of the triangle with vertices $(1,0,-1)$, $(2,3,-2)$, $(0,0,1)$.

Partial Solution: Write $P = (1,0,-1)$, $Q = (2,3,-2)$ and $R = (0,0,1)$. The area of the triangle is then exactly half of the area of the paralellagram that shares any two of the sides. For instance

$$A = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{2}$$

We have

$$\overrightarrow{PQ} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}, \quad PR = -\mathbf{i} + 2\mathbf{k}.$$

After computing the cross-product, we obtain

$$A = \frac{\sqrt{46}}{2}.$$

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(2) (a) [**10 pts**] Consider the two planes given by the equations

$$2x - y + 3z = 6, \quad \text{and} \quad 3x + y - 2z = 10.$$

Find the symmetric equations for the line that lies in the intersection of both planes.

Solution: The planes have normals

$$\mathbf{N}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{N}_2 = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

The direction of the line is then given by

$$\mathbf{L} = \mathbf{N}_1 \times \mathbf{N}_2 = -\mathbf{i} + 13\mathbf{j} + 5\mathbf{k}.$$

Next we need to find a point on the line, namely a point that lies on both planes. We can do this by setting one variable to 0, say $x = 0$ and solving the linear system

$$-y + 3z = 6, \quad y - 2z = 10.$$

This can easily be solved to obtain $y = 42$, $z = 16$. Therefore the equations for the line are

$$-x = \frac{y - 42}{13} = \frac{z - 16}{5}.$$

(b) [**10 pts**] Find the distance from the point $(1, -2, 1)$ to the line with equations

$$\frac{x - 1}{3} = \frac{1 - z}{2}, \quad y = -1.$$

Solution: Let $P_0 = (1, -2, 1)$, and $P_1 = (1, -1, 1)$ be a point on the line. We use the distance formula

$$D = \frac{\|\mathbf{L} \times \overrightarrow{P_0P_1}\|}{\|\mathbf{L}\|}.$$

where

$$\mathbf{L} = 3\mathbf{i} - 2\mathbf{k} \quad \text{and} \quad \overrightarrow{P_0P_1} = \mathbf{j}.$$

You should obtain the answer

$$D = \frac{\sqrt{13}}{\sqrt{13}} = 1.$$

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(3) (a) [**10 pts**] Suppose that the acceleration of an object is given by

$$\mathbf{a}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j},$$

and at time $t = 0$, we know $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{0}$. Find the formula for the position $\mathbf{r}(t)$ of the object.

Solution: The position $\mathbf{r}(t)$ satisfies

$$\mathbf{r}''(t) = \mathbf{a}(t).$$

Integrating the equation twice and using the fundamental theorem of calculus for vector-valued functions gives

$$\mathbf{r}'(t) = \sin t\mathbf{i} - 2\cos t\mathbf{j} + \mathbf{C}_0,$$

and

$$\mathbf{r}(t) = -\cos t\mathbf{i} - 2\sin t\mathbf{j} + \mathbf{C}_0 t + \mathbf{C}_1.$$

Using the fact that $\mathbf{r}'(0) = \mathbf{v}(0) = \mathbf{0}$ tells us that $\mathbf{C}_0 = 2\mathbf{j}$ and similarly $\mathbf{r}(0) = \mathbf{0}$ implies that $\mathbf{C}_1 = \mathbf{i}$. The find answer is then given by

$$\mathbf{r}(t) = (1 - \cos t)\mathbf{i} + 2(t - \sin t)\mathbf{j}.$$

(b) **[10 pts]** Suppose the position of an object is given by the vector-valued function $\mathbf{r}(t)$, and suppose that it's accelleration $\mathbf{a}(t)$ is paralell to $\mathbf{r}(t)$. Show that $\mathbf{r}(t) \times \mathbf{v}(t)$ is constant.

Solution: Following the hint given in class. We take the derivative of $\mathbf{r}(t) \times \mathbf{v}(t)$,

$$(\mathbf{r}(t) \times \mathbf{v})'(t) = \underbrace{\mathbf{v}(t) \times \mathbf{v}(t)}_{=0} + \underbrace{\mathbf{r}(t) \times \mathbf{a}(t)}_{=0} = 0.$$

The first term is zero by properties of the cross-product. The second term is zero since we assumed that $\mathbf{a}(t)$ is paralell to $\mathbf{r}(t)$.

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(4) **[20 pts]** Consider the vector object with position given by

$$\mathbf{r}(t) = t\mathbf{i} + \frac{3}{2}\sqrt{2}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}.$$

Find $a_{\mathbf{T}}(t)$ and $a_{\mathbf{N}}(t)$. **Assigned as extra assignment**

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(5) (a) **[15 pts]** Let

$$\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + 2\mathbf{k}.$$

Find the length of the curve parameterized by $\mathbf{r}(t)$ for $0 \leq t \leq 1$.

Solution: The length of the curve is given by

$$L = \int_0^1 \|\mathbf{r}'(t)\|.$$

Using the fact that

$$\mathbf{r}'(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\sin t + \cos t)\mathbf{j}.$$

and

$$\|\mathbf{r}'(t)\| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = e^t\sqrt{2}.$$

The integral becomes

$$L = \sqrt{2}\int_0^1 e^t = \sqrt{2}(e^1 - 1).$$

(b) **[5 pts]** Determine whether the function

$$\mathbf{r}(t) = (t^2 - 6t)\mathbf{i} + (t-3)^4\mathbf{j} + t(t^2 - 27)\mathbf{k}$$

is smooth or piecewise smooth.

Solution: The function $\mathbf{r}(t)$ is clearly a differentialble function of t since each of it's components are just polynomials. To see whether it is a smooth parameterization, take the derivative,

$$\mathbf{r}'(t) = (2t - 6)\mathbf{i} + 4(t-3)^3\mathbf{j} + 3t^2 - 27\mathbf{k}.$$

We then check to see if $\mathbf{r}'(t)$ is zero for any t. Indeed, we find that $\mathbf{r}'(3) = 0$ so that the parameterization is only piecewise smooth.