Math 241 Exam 2 Fall 2014 Partial Solutions **Justin Wyss-Gallifent**

1. (a) Use the gradient to find a vector perpendicular to the graph of the curve $y = x^3 + x - 2$ at the point where $x = 2$. [10 pts]

   Partial Solutions: Move all the variables to one side and let that side be the function $f(x, y)$. Then find ∇f at $(2, 8)$ where the 8 comes from plugging 2 into $y = x^3 + x - 2$.

   (b) Suppose the base of a triangle is growing at 2 inches per hour while the height is growing at 3 inches per hour. At what rate is the area growing when the height is 10 inches and the base is 20 inches? [10 pts]

   Partial Solutions: Since $A = \frac{1}{2}bh$ we can apply the chain rule since b and h are functions of t.

2. (a) Sketch the graph of the surface $y^2 = x^2 + z^2$. Write the name. [5 pts]

   Partial Solutions: Sketch omitted - this is a double-cone opening both directions around the y-axis with the vertex at the origin.

   (b) Sketch the graph of the surface $y = x^2$. Write the name. [5 pts]

   Partial Solutions: Sketch omitted - this is a parabolic sheet - take $y = x^2$ in the xy-plane and extend it in the z-direction.

   (c) Find the directional derivative of $f(x, y) = y \sin(xy)$ in the direction of $\bar{a} = 2\,\hat{\imath} + \,\hat{\jmath}$ at the point $\left(\frac{\pi}{8}, 2\right)$. Simplify. [10 pts]

   Partial Solutions: First make $\bar{a}$ a unit vector then simply apply the formula for the directional derivative.

3. (a) All together on one graph sketch the level curves for $f(x, y) = y - |x|$ at $c = -2, 0, 2$ and label each with its value of c. [5 pts]

   Partial Solutions: Sketches omitted - The curves are $y = |x| - 2$, $y = |x|$ and $y = |x| + 2$. These are all absolute value vertical shifts.

   (b) Suppose the unit vector $\bar{u}$ makes an angle of $30°$ with the gradient of a function f at $(1, 2)$ and $||\nabla f(1, 2)|| = 3$. Find $D_{\bar{u}} f(1, 2)$. [5 pts]

   Partial Solutions: Remember that $D_{\bar{u}} f(1, 2) = \bar{u} \cdot \nabla f(1, 2) = ||\bar{u}||||\nabla f(1, 2)|| \cos\theta$. Then $\bar{u}$ is a unit vector and plug the rest in.

   (c) The function $f(x, y) = x^2 y - 2x^2 - y^2$ has the following: [10 pts]

   $$f_{xx}(x, y) = 2y - 4 \qquad f_{yy}(x, y) = -2 \qquad f_{xy}(x, y) = 2x$$

   There are three critical points at $(0, 0)$, $(2, 2)$ and $(-2, 2)$. Categorize each critical point as a relative maximum, relative minimum or saddle point.

   Partial Solutions: This is pretty straightforward since everything is given. Don't get the discriminant incorrect!

4. Find the maximum and minimum values of $f(x, y) = x^2 + 2y^2$ on the quarter circle $x^2 + y^2 \leq 4$ with $x, y \geq 0$. [20 pts]

Partial Solutions: The critical point is at $(0, 0)$ which is on the edge and $f(0, 0) = 0$. The edge is composed of three pieces:

- The quarter circle $x^2 + y^2 = 4$ where $y^2 = 4 - x^2$ and so $f = x^2 + 2(4 - x^2) = 8 - x^2$ with $0 \leq x \leq 2$. What's the max and min?
- The lower line segment (on the x-axis) which is $y = 0$ and so $f = x^2$ with $0 \leq x \leq 2$. What's the max and min?
- The left line segment (on the y-axis) which is $x = 0$ and so $f = 2y^2$ with $0 \leq y \leq 2$. What's the max and min?

Pick the max and min from among them all.

5. Let $f(x, y) = x^2 + 6y^2$ and suppose (x, y) is constrained by $x + 3y = 10$.

   (a) Use Lagrange multipliers to find the minimum of $f(x, y)$ subject to the constraint. [16 pts]
   Partial Solutions: The system is:

$$\begin{aligned} 2x &= \lambda(1) \\ 12y &= \lambda(3) \\ x + 3y &= 10 \end{aligned}$$

   This has only one solution at $(4, 2)$.

   (b) Explain why $f(x, y)$ has no maximum subject to the constraint. [4 pts]
   Partial Solutions: Because on the constraint $x = 10 - 3y$ and so $f(x, y) = x^2 + 6y^2 = (10 - 3y)^2 + 6y^2 = 100 - 60y + 15y^2$ which can be made as large as we want by making y large.