Math 241 - Exam 2 -Solution

Tuesday, Oct 17th, 2017

You have 50 minutes to complete this exam. Do not simplify unless indicated. Calculators are **not** allowed. Submit each of the five problems on a separate sheet. Please cross out any work you don't want to be graded.

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(1) **[20 pts]** For this problem, consider the surface

$$\frac{x^2}{2} + \frac{y^2}{4} - 2z = 6.$$

(a) **[12 pts]**Find the point (x_0, y_0, z_0) on the surface with normal vector paralell to $3\mathbf{i} + \mathbf{j} - \mathbf{k}$.
Be sure that the point you find actually lies on the surface!

Solution: A normal to the surface at (x, y, z) is

$$\mathbf{N} = x\mathbf{i} + \frac{1}{2}y\mathbf{j} - 2\mathbf{k}.$$

This will be parallel to $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ if there is a c such that

$$x = 3c, \quad \frac{1}{2}y = c, \quad -2 = -c.$$

Solving this for x and y gives $x = 6$, $y = 4$. To determine the z value, we plug this into the equation for the surface and obtain

$$z = \frac{1}{2}\left(\frac{36}{2} + \frac{16}{4} - 6\right) = 8.$$

Therefore the point $(6, 4, 8)$ is the one we are looking for.

(b) **[8 pts]** Find the equation for the plane tangent to the surface at the point $(2, 4, 0)$.

Solution: At $(2, 4, 0)$, a normal vector to the plane is $\mathbf{N} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Therefore the equation for the tangent plane is

$$2(x - 2) + 2(y - 4) - 2z = 0.$$

This simplifies to

$$x + y - z = 6.$$

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(2) **[20 pts]** Let

$$f(x, y, z) = \sqrt{5x^2 + 3y^2 + 2z^2}.$$

Find an approximation of $f(0.9, 2.1, 2.1)$ using the tangent 'plane' approximation. Simplify your answer. *You should not need a calculator.*

Solution: $(0.9, 2.1, 2.1)$ is close to $(1, 2, 2)$, and it is easy to verify that

$$f(1, 2, 2) = \sqrt{25} = 5.$$

The partial derivatives of f are

$$f_x = \frac{5x}{\sqrt{5x^2+3y^2+2z^2}}, \quad f_y = \frac{3y}{\sqrt{5x^2+3y^2+2z^2}}, \quad f_z = \frac{2z}{\sqrt{5x^2+3y^2+2z^2}}.$$

when evaluated at $(1, 2, 2)$ give

$$f_x(1,2,2) = 1, \quad f_y(1,2,2) = \frac{6}{5}, \quad f_z(1,2,2) = \frac{4}{5}.$$

The tangent plane approximation is then given by

$$\begin{aligned} f(.9, 2.1, 2.1) &\approx f(1,2,2) + f_x(1,2,2)(.9-1) + f_y(1,2,2)(2.1-2) + f_z(1,2,2)(2.1-2) \\ &= 5 + 1(-.1) + \frac{6}{5}(.1) + \frac{4}{5}(.1) \\ &= 5.1. \end{aligned}$$

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(3) [**20 pts**] For this problem, consider the function

$$f(x, y) = \sin(xy) - xy.$$

(a) [**10 pts**] Find the unit vector direction in which f *increases* the most rapidly at the point $(1, \pi/2)$. What is the the largest directional derivative at this point?

Solution: The gradient of f is

$$\nabla f(x, y) = (y \cos xy - y)\,\mathbf{i} + (x \cos xy - x)\,\mathbf{j},$$

and gives the direction of greatest increas of f at (x, y). Evaluating this at $(1, \pi/2)$ gives

$$\begin{aligned} \nabla f(1, \pi/2) &= \left(\frac{\pi}{2}\cos\frac{\pi}{2} - \frac{\pi}{2}\right)\mathbf{i} + \left(\cos\frac{\pi}{2} - 1\right)\mathbf{j} \\ &= -\frac{\pi}{2}\mathbf{i} - \mathbf{j}. \end{aligned}$$

Normalizing this gives the unit vector of greatest increase

$$\mathbf{u} = \frac{-\pi}{\sqrt{\pi^2+4}}\mathbf{i} + \frac{-2}{\sqrt{\pi^2+4}}\mathbf{j}.$$

The largest directional derivative is then just given by

$$D_{\mathbf{u}} f(1, \pi/2) = \|\nabla f(1, \pi/2)\| = \sqrt{\left(\frac{\pi}{2}\right)^2 + 1}.$$

(b) [**10 pts**] Find the unit vector direction in which f *decreases* the most at the point $(1, \pi)$.

Solution: The direction of greatest decrease of the f is in the diretion of the negative of the gradient. This gives the following unit vector

$$\mathbf{u} = \frac{-\nabla f(1, \pi)}{\|\nabla f(1, \pi)\|} = \frac{\pi}{\sqrt{\pi^2+1}}\mathbf{i} + \frac{1}{\sqrt{\pi^2+1}}\mathbf{j}.$$

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(4) **[20 pts]** Find and classify all critical points of the function

$$f(x,y) = 2x^2y - 4x^2 - 2y^2.$$

as a relative minimum, relative maximum, or saddle.

Solution: The critical points are located at (x,y) which satisfy

$$f_x = 4xy - 8x = 0, \quad f_y = 2x^2 - 4y = 0.$$

The first equation has solutions at $x = 0$ and $y = 2$, substituting this into the second equation implies that when $x = 0$, then $y = 0$ and when $y = 2$ then $x = \pm 2$. Therefore the critical points are at

$$(0,0), \quad (2,2), \quad (-2,2).$$

To classify these points we conduct the second derivative test. We have

$$f_{xx} = 4y - 8, \quad f_{yy} = -4, \quad f_{xy} = 4x,$$

and so

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = -16(y-2) - (4x)^2.$$

Since

$$D(0,0) = 32 > 0 \quad f_{xx}(0,0) = -8 < 0,$$

then $(0,0)$ is a relative min. Also since

$$D(\pm 2, 2) = -64 < 0,$$

then both $(2,2)$ and $(-2,2)$ are saddles.

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(5) **[20 pts]** Consider the function

$$f(x,y) = 2xy.$$

Use Lagrange multipliers to find the global extreme values of of $f(x,y)$ on the ellipse

$$x^2 + 2y^2 = 4.$$

Hint: there should be 4 critical points for the constrained system.

Solution: The constraing function is $g(x,y) = x^2 + 2y^2$. Using Lagrange multipliers, we must find solutions of the system

$$2y = \lambda 2x, \quad 2x = \lambda 4y, \quad x^2 + 2y^2 = 4.$$

Solving for λ in the first two equations and setting the expressions equal (eliminating λ) gives

$$\frac{y}{x} = \frac{x}{2y}.$$

This implies that $2y^2 = x^2$. Substituting this into the constraint then gives,

$$2x^2 = 4,$$

and therefore $x = \pm\sqrt{2}$. For each of these x values the constraint implies that y has to solve

$$y^2 = 1,$$

and therefore $y = \pm 1$. This implies that there are four critical points given by

$$(\sqrt{2}, 1), \quad (-\sqrt{2}, 1), \quad (\sqrt{2}, -1), \quad (-\sqrt{2}, -1).$$

Substituting these values back into f gives

$$\begin{aligned} f(\sqrt{2}, 1) &= 2\sqrt{2} \\ f(-\sqrt{2}, 1) &= -2\sqrt{2} \\ f(\sqrt{2}, -1) &= -2\sqrt{2} \\ f(-\sqrt{2}, -1) &= 2\sqrt{2}. \end{aligned}$$

It follows that there is a global max of $2\sqrt{2}$ at the points $(\sqrt{2}, 1)$ and $(-\sqrt{2}, -1)$ and a global min of $-2\sqrt{2}$ at the points $(-\sqrt{2}, 1)$ and $(\sqrt{2}, -1)$.